ALMOST ALL OPTIMALLY COLOURED COMPLETE GRAPHS CONTAIN A RAINBOW HAMILTON PATH

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ABSTRACT. A subgraph H of an edge-coloured graph is called rainbow if all of the edges of H have different colours. In 1989, Andersen conjectured that every proper edge-colouring of K_n admits a rainbow path of length $n-2$. We show that almost all optimal edge-colourings of K_n admit both (i) a rainbow Hamilton path and (ii) a rainbow cycle using all of the colours. This result demonstrates that Andersen's Conjecture holds for almost all optimal edge-colourings of K_n and answers a recent question of Ferber, Jain, and Sudakov. Our result also has applications to the existence of transversals in random symmetric Latin squares.

1. INTRODUCTION

1.1. **Extremal results on rainbow colourings.** We say that a subgraph H of an edge-coloured graph is *rainbow* if all of the edges of H have different colours. An *optimal edge-colouring* of a graph is a proper edge-colouring using the minimum possible number of colours. In this paper we study the problem of finding a rainbow Hamilton path in large optimally edge-coloured complete graphs.

The study of finding rainbow structures within edge-coloured graphs has a rich history. For example, the problem posed by Euler on finding orthogonal $n \times n$ Latin squares can easily be seen to be equivalent to that of finding an optimal edge-colouring of the complete bipartite graph $K_{n,n}$ which decomposes into edge-disjoint rainbow perfect matchings. It transpires that there are optimal colourings of $K_{n,n}$ without even a single rainbow perfect matching, if n is even. However, an important conjecture, often referred to as the Ryser-Brualdi-Stein Conjecture, posits that one can always find an almost-perfect rainbow matching, as follows.

Conjecture 1.1 (Ryser [36], Brualdi-Stein [9,37]). *Every optimal edge-colouring of $K_{n,n}$ admits a rainbow matching of size $n-1$ and, if n is odd, a rainbow perfect matching.*

Currently, the strongest result towards this conjecture for arbitrary optimal edge-colourings is due to Keevash, Pokrovskiy, Sudakov, and Yepremyan [24], who showed that there is always a rainbow matching of size $n - O(\log n / \log \log n)$. This result improved earlier bounds of Woolbright [38], Brouwer, de Vries, and Wieringa [7], and Hatami and Shor [21].

It is natural to search for spanning rainbow structures in the non-partite setting as well; that is, what spanning rainbow substructures can be found in properly edge-coloured complete graphs K_n? It is clear that one can always find a rainbow spanning tree – indeed, simply take the star rooted at any vertex. Kaneko, Kano, and Suzuki [22] conjectured that for $n > 4$, in any proper edge-colouring of K_n, one can find $\lfloor n/2 \rfloor$ edge-disjoint rainbow spanning trees, thus decomposing K_n if n is even, and almost decomposing K_n if n is odd. This conjecture was recently proved approximately by Montgomery, Pokrovskiy, and Sudakov [32], who showed that in any properly edge-coloured K_n, one can find $(1 - o(1))n/2$ edge-disjoint rainbow spanning trees.

For *optimal* edge-colourings, even more is known. Note firstly that if n is even and K_n is optimally edge-coloured, then the colour classes form a *1-factorization* of K_n; that is, a decomposition of K_n into perfect matchings. Throughout the paper, we will use the term 1-factorization synonymously with an edge-colouring whose colour classes form a 1-factorization.

It is clear that if a 1-factorization of K_n exists, then n is even. Very recently, Glock, Kühn, Montgomery, and Osthus [16] showed that for sufficiently large even n, there exists a tree T on n vertices such that any 1-factorization of K_n decomposes into edge-disjoint rainbow spanning trees isomorphic to T, thus resolving conjectures of Brualdi and Hollingsworth [8], and Constantine [11, 12]. See e.g. [25, 32, 34] for previous work on these conjectures.

The tree T used in [16] is a path of length $n - o(n)$, together with $o(n)$ short paths attached to it. Thus it might seem natural to ask if one can find a rainbow Hamilton path in any 1-factorization of K_n. Note that such a path would contain all of the colours used in the 1-factorization, so it is not possible to find a rainbow Hamilton cycle in a 1-factorization of K_n. However, in 1984 Maamoun and Meyniel [29] proved the existence of a 1-factorization of K_n (for $n \geq 4$ being any power of 2) without a rainbow Hamilton path. Sharing parallels with Conjecture 1.1 for the non-partite setting, Andersen [5] conjectured in 1989 that all proper edge-colourings of K_n admit a rainbow path which omits only one vertex.

Conjecture 1.2 (Andersen [5]). *All proper edge-colourings of K_n admit a rainbow path of length $n - 2$.*

Several variations of Andersen's Conjecture have been proposed. In 2007, Akbari, Etesami, Mahini, and Mahmoody [1] conjectured that all 1-factorizations of K_n admit a Hamilton cycle whose edges collectively have at least $n - 2$ colours. They also conjectured that all 1-factorizations of K_n admit a rainbow cycle omitting only two vertices.

Although now known to be false, the following stronger form of Conjecture 1.2 involving the "sub-Ramsey number" of the Hamilton path was proposed by Hahn [19]. Every (not necessarily proper) edge-colouring of K_n with at most $n/2$ edges of each colour admits a rainbow Hamilton path. In light of the aforementioned construction of Maamoun and Meyniel [29], in 1986 Hahn and Thomassen [20] suggested the following slightly weaker form of Hahn's Conjecture, that all edge-colourings of K_n with strictly fewer than $n/2$ edges of each colour admit a rainbow Hamilton path. However, even this weakening of Hahn's Conjecture is false – Pokrovskiy and Sudakov [35] proved the existence of such edge-colourings of K_n in which the longest rainbow Hamilton path has length at most $n - \ln n/42$.

Andersen's Conjecture has led to a number of results, generally focussing on increasing the length of the rainbow path or cycle that one can find in an arbitrary 1-factorization or proper edge-colouring of K_n (see e.g. [10, 15, 17, 18]). Alon, Pokrovskiy, and Sudakov [3] proved that all proper edge-colourings of K_n admit a rainbow path with length $n - O(n^{3/4})$, and the error bound has since been improved to $O(\sqrt{n} \cdot \log n)$ by Balogh and Molla [6]. Further support for Conjecture 1.2 and its variants was provided by Montgomery, Pokrovskiy, and Sudakov [32] as well as Kim, Kühn, Kupavskii, and Osthus [25], who showed that if we consider proper edge-colourings where no colour class is larger than $n/2 - o(n)$, then we can even find $n/2 - o(n)$ edge-disjoint rainbow Hamilton cycles.

1.2. **Random colourings.** It is natural to consider these problems in a probabilistic setting, that is to consider random edge-colourings as well as random Latin squares. However, the "rigidity" of the underlying structure makes these probability spaces very challenging to analyze. Recently significant progress was made by Kwan [26], who showed that almost all Latin squares contain a transversal, or equivalently, that almost all optimal edge-colourings of $K_{n,n}$ admit a rainbow perfect matching. His analysis was carried out in a hypergraph setting, which also yields the result that almost all Steiner triple systems contain a perfect matching. Recently, this latter result was strengthened by Ferber and Kwan [14], who showed that almost all Steiner triple systems have an approximate decomposition into edge-disjoint perfect matchings. Here we show that Hahn's original conjecture (and thus Andersen's Conjecture as well) holds for almost all 1-factorizations, answering a recent question of Ferber, Jain, and Sudakov [13]. In what follows, we say a property holds 'with high probability' if it holds with a probability that tends to 1 as the number of vertices n tends to infinity.

Theorem 1.3. *Let ϕ be a uniformly random optimal edge-colouring of K_n. Then with high probability,*

(i) ϕ admits a rainbow Hamilton path, and

(ii) ϕ admits a rainbow cycle F containing all of the colours.

In particular, if n is odd, then F is a rainbow Hamilton cycle.

As discussed in Section 7, there is a well-known correspondence between rainbow 2-factors in n-edge-colourings of K_n and transversals in symmetric Latin squares, as a transversal in a Latin square corresponds to a permutation σ of $[n]$ such that the entries in positions $(i, \sigma(i))$ are distinct for all $i \in [n]$. Based on this, we use Theorem 1.3(ii) to show that random symmetric Latin squares of odd order contain a *Hamilton transversal* with high probability. Here we say a transversal is *Hamilton* if the underlying permutation σ is an n-cycle.

Corollary 1.4. *Let n be an odd integer and $\mathbf{L}$ a uniformly random symmetric $n \times n$ Latin square. Then with high probability $\mathbf{L}$ contains a Hamilton transversal.*

Further results on random Latin squares were recently obtained by Kwan and Sudakov [27], who gave estimates on the number of intercalates in a random Latin square as well as their likely discrepancy.

2. Notation

In this section, we collect some definitions and notation that we will use throughout the paper.

For a graph G and (not necessarily distinct) vertex sets $A, B \subseteq V(G)$, we define $E_G(A, B) \coloneqq \{e = ab \in E(G) : a \in A, b \in B\}$. We often simply write $E(A, B)$ when G is clear from the context. We define $e(A, B) \coloneqq |E(A, B)|$. For a vertex $v \in V(G)$, we define $\partial_G(v)$ to be the set of edges of G which are incident to v. For a proper colouring $\phi \colon E(G) \to \mathbb{N}$ and a colour $c \in \mathbb{N}$, we define $E_c(G) \coloneqq \{e \in E(G) : \phi(e) = c\}$ and say that an edge $e \in E_c(G)$ is a *c-edge* of G. For a vertex $v \in V(G)$, if e is a c-edge in G incident to v, then we say that the non-v endpoint of e is the *c-neighbour* of v. For a vertex $v \in V(G)$ and three colours $c_1, c_2, c_3 \in \mathbb{N}$, we say that the c_3-neighbour of the c_2-neighbour of the c_1-neighbour of v is the *end* of the *$c_1 c_2 c_3$-walk starting at v*, if all such edges exist. For a set of colours $D \subseteq \mathbb{N}$, we define $N_D(v) \coloneqq \{w \in N_G(v) : \phi(vw) \in D\}$. For sets $A, B \subseteq V(G)$ and a colour $c \in \mathbb{N}$, we define $E_c(A, B) \coloneqq \{e \in E(A, B) : \phi(e) = c\}$. If G is not clear from the context, we sometimes also write $E_G^c(A, B)$. For any subgraph $H \subseteq G$, we define $\phi(H) \coloneqq \{\phi(e) : e \in E(H)\}$. For a set of colours $D \subseteq [n-1]$, let $\mathcal{G}_D^{\mathrm{col}}$ be the set of pairs (G, ϕ_G), where G is a $|D|$-regular graph on a vertex set V of size n, and ϕ_G is a 1-factorization of G with colour set D. Often, we abuse notation and write $G \in \mathcal{G}_D^{\mathrm{col}}$, and in this case we let ϕ_G denote the implicit 1-factorization of G, sometimes simply writing ϕ when G is clear from the context. A subgraph $H \subseteq G \in \mathcal{G}_D^{\mathrm{col}}$ inherits the colours of its edges from G. Observe that uniformly randomly choosing a 1-factorization ϕ of K_n on vertex set V and colour set $[n-1]$ is equivalent to uniformly randomly choosing $G \in \mathcal{G}_{[n-1]}^{\mathrm{col}}$. For any $D \subseteq [n-1]$, $G \in \mathcal{G}_D^{\mathrm{col}}$, and sets $V' \subseteq V$, $D' \subseteq D$, we define $E_{V', D'}(G) \coloneqq \{e = xy \in E(G) : \phi_G(e) \in D', x, y \in V'\}$, and we define $e_{V', D'}(G) \coloneqq |E_{V', D'}(G)|$. For a hypergraph $\mathcal{H}$, we write $\Delta^c(\mathcal{H})$ to denote the maximum codegree of $\mathcal{H}$; that is, the maximum number of edges containing any two fixed vertices of $\mathcal{H}$.

For a set D of size n and a partition $\mathcal{P}$ of D into m parts, we say that $\mathcal{P}$ is *equitable* to mean that all parts P of $\mathcal{P}$ satisfy $|P| \in \{\lfloor n/m \rfloor, \lceil n/m \rceil\}$, and when it does not affect the argument, we will assume all parts of an equitable partition have size precisely n/m. For a set S and a real number $p \in [0, 1]$, a *p-random* subset $T \subseteq S$ is a random subset in which each element of S is included in T independently with probability p. A *β-random* subgraph of a graph G is a spanning subgraph of G where the edge-set is a β-random subset of $E(G)$. For an event $\mathcal{E}$ in any probability space, we write $\overline{\mathcal{E}}$ to denote the complement of $\mathcal{E}$. For real numbers a, b, c such that $b > 0$, we write $a = (1 \pm b)c$ to mean that the inequality $(1 - b)c \leq a \leq (1 + b)c$ holds. For a natural number $n \in \mathbb{N}$, we define $[n] \coloneqq \{1, 2, \ldots, n\}$, and $[n]_0 \coloneqq [n] \cup \{0\}$. We write $x \ll y$ to mean that for any $y \in (0, 1]$ there exists an $x_0 \in (0, 1)$ such that for all $0 < x \leq x_0$ the subsequent statement holds. Hierarchies with more constants are defined similarly and should be read from the right to the left. Constants in hierarchies will always be real numbers in $(0, 1]$. We assume large numbers to be integers if this does not affect the argument.

3. Proving Theorem 1.3

In this section, let ϕ be a 1-factorization of K_n with vertex set V and colour set $C = [n-1]$. We provide an overview of the proof of Theorem 1.3, and in fact, we prove Theorem 1.3 in the case when n is even in Section 3.2 assuming two key lemmas which we prove in later sections. In particular, we assume that n is even in Sections 3–6, so that the optimal edge-colouring ϕ we work with is a 1-factorization of K_n. In Section 7 we derive Theorem 1.3 in the case when n is odd from the case when n is even. We will also deduce Corollary 1.4 from Theorem 1.3(ii) in Section 7. Throughout the proof we work with constants $\varepsilon, \gamma, \eta$, and μ satisfying the following hierarchy:

$$(3.1) \qquad\qquad 1/n \ll \varepsilon \ll \gamma \ll \eta \ll \mu \ll 1,$$

Our proof uses the absorption method as well as switching techniques. Note that the latter is a significant difference to [14, 26], which rely on the analysis of the random greedy triangle removal process, as well as modifications of arguments in [23, 28] which bound the number of Steiner triple systems. Our main objective is to show that with high probability, in a random 1-factorization, we can find an absorbing structure inside a random subset of $\Theta(\mu n)$ reserved vertices, using a random subset of $\Theta(\mu n)$ reserved colours. A recent result [16, Lemma 16], based on hypergraph matchings, enables us to find a long rainbow path avoiding these reserved vertices and colours, and using our absorbing structure, we extend this path to a rainbow Hamilton path. More precisely, we randomly 'reserve' $\Theta(\mu n)$ vertices and colours and show that with high probability we can find an absorbing structure. This absorbing structure consists of a subgraph G_{abs} containing only reserved vertices and colours and all but at most γn of them. Moreover G_{abs} contains 'flexible' sets of vertices and colours V_{flex} and C_{flex} each of size ηn, with the following crucial property:

> (†) for any pair of equal-sized subsets $X \subseteq V_{\mathrm{flex}}$ and $Y \subseteq C_{\mathrm{flex}}$ of size at most $\eta n/2$, the graph $G_{\mathrm{abs}} - X$ contains a spanning rainbow path whose colours avoid Y.

Given this absorbing structure, we find a rainbow Hamilton path in the following three steps:

> (1) **Long path step:** Apply [16, Lemma 16] to obtain a long rainbow path P_1 containing only non-reserved vertices and colours and all but at most γn of them.
>
> (2) **Covering step:** 'Cover' the vertices and colours not in G_{abs} or P_1 using the flexible sets, by greedily constructing a path P_2 containing them as well as sets $X \subseteq V_{\mathrm{flex}}$ and $Y \subseteq C_{\mathrm{flex}}$ of size at most $\eta n/2$.
>
> (3) **Absorbing step:** 'Absorb' the remaining vertices and colours, by letting P_3 be the rainbow path guaranteed by (†).

In the covering step, we can ensure that P_2 shares one end with P_1 and one end with P_3 so that $P_1 \cup P_2 \cup P_3$ is a rainbow Hamilton path, as desired. These steps are fairly straightforward, so the majority of the paper is devoted to building the absorbing structure.

In Section 3.1, we present the details of this absorbing structure, and in Section 3.2, we prove Theorem 1.3 (in the case when n is even) subject to its existence.

3.1. Absorption.
Our absorbing structure consists of many 'gadgets' pieced together in a particular way. First, we introduce these gadgets in the following definition (see also Figure 1).

Definition 3.1. For every $v \in V$ and $c \in C$, a *(v, c)-absorbing gadget* is a subgraph of K_n of the form $A = T \cup Q$ such that the following holds:

- $T \cong K_3$ and $Q \cong C_4$,
- T and Q are vertex-disjoint,
- $v \in V(T)$ and there is a unique edge $e \in E(Q)$ such that $\phi(e) = c$,
- if $e_1, e_2 \in E(T)$ are the edges incident to v, then there is matching $\{e_1', e_2'\}$ in Q not containing e such that $\phi(e_i) = \phi(e_i')$ for $i \in \{1, 2\}$,
- if $e_3 \in E(T)$ is the edge not incident to v, then there is an edge $e_3' \in E(Q)$ such that $\{e_3', e\}$ is a matching in Q and $\phi(e_3) = \phi(e_3') \neq c$.

In this case, a pair P_1, P_2 of paths *completes* the (v, c)-absorbing gadget $A = T \cup Q$ if

- the ends of P_1 are non-adjacent vertices in Q,
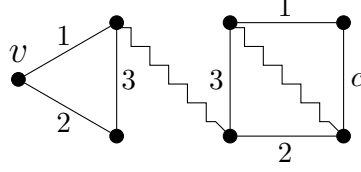- one end of P_2 is in Q but not incident to e and the other end of P_2 is in $V(T) \setminus \{v\}$,

FIGURE 1. A (v,c)-absorber, where $\phi(e_i) = i$. The paths P_1 and P_2 are drawn as zigzags.

- P_1 and P_2 are vertex-disjoint and both P_1 and P_2 are internally vertex-disjoint from A,
- $P_1 \cup P_2$ is rainbow, and
- $\phi(P_1 \cup P_2) \cap \phi(A) = \varnothing$,

and we say $A' := A \cup P_1 \cup P_2$ is a (v,c)-absorber. We also define the following.
- The path P with edge-set $E(P_1) \cup E(P_2) \cup \{e'_1, e'_2, e_3\}$ is the (v,c)-avoiding path in A', and the path P' with edge-set $E(P_1) \cup E(P_2) \cup \{e_1, e_2, e'_3, e\}$ is the (v,c)-absorbing path in A'.
- A vertex in $V(A) \setminus \{v\}$, a colour in $\phi(A) \setminus \{c\}$, or an edge in $E(A)$ is used by the (v,c)-absorbing gadget A.

Suppose A is a (v,c)-absorber that is part of the absorbing structure G_{abs}. Importantly, the (v,c)-absorbing path is rainbow, spanning in A, and contains all the colours in $\phi(A)$, and the (v,c)-avoiding path is rainbow, spanning in $A - v$, and contains all the colours in $\phi(A) \setminus \{c\}$. Thus, when proving (†), if v or c is in X or Y, then the spanning rainbow path in $G_{\text{abs}} - X$ contains the (v,c)-avoiding path, and otherwise it may contain the (v,c)-absorbing path.

We build our absorbing structure out of (v,c)-absorbers, using a bipartite graph H as a template, where one part of H is a subset of V and the other part is a subset of C. For every edge $vc \in E(H)$, we will have a (v,c)-absorber in the absorbing structure. It is convenient for us to distinguish between a (v,c)-absorbing gadget and a (v,c)-absorber, because when we build our absorbing structure, we first find a (v,c)-absorbing gadget for every $vc \in E(H)$ and then find the paths completing each absorbing gadget, as in the following definition.

Definition 3.2. Let H be a bipartite graph with bipartition (V', C') where $V' \subseteq V$ and $C' \subseteq C$, and suppose $\mathcal{A} = \{A_{v,c} : vc \in E(H)\}$ where $A_{v,c}$ is a (v,c)-absorbing gadget.
- We say $\mathcal{A}$ *satisfies* H if whenever $A_{v,c}, A_{v',c'} \in \mathcal{A}$ for some $(v,c) \neq (v',c')$, no vertex in $V(A_{v,c})$ or colour in $\phi(A_{v,c})$ is used by $A_{v',c'}$.
- If $\mathcal{P}$ is a collection of vertex-disjoint paths of length 4, then we say $\mathcal{P}$ *completes* $\mathcal{A}$ if the following holds:
    - $\bigcup_{P \in \mathcal{P}} P$ is rainbow,
    - no colour that is either in C' or is used by a (v,c)-absorbing gadget $A_{v,c} \in \mathcal{A}$ appears in a path $P \in \mathcal{P}$,
    - no vertex that is either in V' or is used by a (v,c)-absorbing gadget $A_{v,c} \in \mathcal{A}$ is an internal vertex of a path $P \in \mathcal{P}$,
    - for every (v,c)-absorbing gadget $A_{v,c} \in \mathcal{A}$ there is a pair of paths $P_1, P_2 \in \mathcal{P}$ such that P_1 and P_2 complete $A_{v,c}$ to a (v,c)-absorber $A'_{v,c}$, and
    - the graph $\left(\bigcup_{A \in \mathcal{A}} A \cup \bigcup_{P \in \mathcal{P}} P\right) \setminus V'$ is connected and has maximum degree three, and $\mathcal{P}$ is minimal subject to this property.
- We say $(\mathcal{A}, \mathcal{P})$ is an *H-absorber* if $\mathcal{A}$ satisfies H and is completed by $\mathcal{P}$. See Figure 2.
- We say a rainbow path T is a *tail* of an H-absorber $(\mathcal{A}, \mathcal{P})$ if
    - one of the ends of T, say x, is in a (v,c)-absorbing gadget $A_{v,c} \in \mathcal{A}$ such that $x \neq v$,
    - $V(T) \cap V(A) \subseteq \{x\}$ for all $A \in \mathcal{A}$ and $V(T) \cap V(P) = \varnothing$ for all $P \in \mathcal{P}$, and
    - $\phi(T) \cap \phi(A) = \varnothing$ for all $A \in \mathcal{A}$ and $\phi(T) \cap \phi(P) = \varnothing$ for all $P \in \mathcal{P}$.
- For every matching M in H, we define the *path absorbing M* in $(\mathcal{A}, \mathcal{P}, T)$ to be the rainbow path P such that
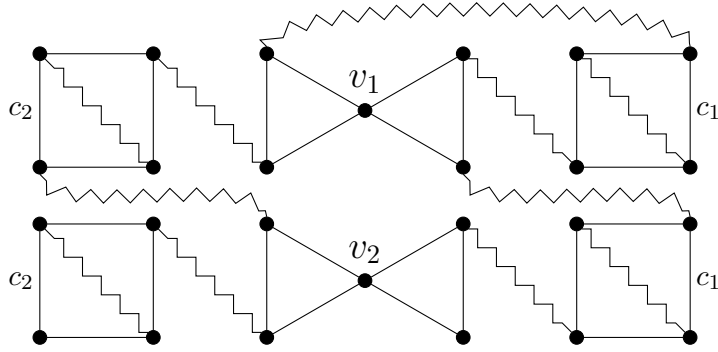   - P contains $\bigcup_{P' \in \mathcal{P}} P' \cup T$ and

FIGURE 2. An H-absorber where $H \cong K_{2,2}$ with bipartition $(\{v_1, v_2\}, \{c_1, c_2\})$.

– for every $vc \in E(H)$, if $vc \in E(M)$, then P contains the (v, c)-absorbing path in the (v, c)-absorber $A'_{v,c}$ and P contains the (v, c)-avoiding path otherwise (that is, $V(P) \cap V' = V(M) \cap V'$ and $\phi(P) \cap C' = V(M) \cap C'$).

Note that if $\mathcal{P}$ completes $\mathcal{A}$, then some of the paths in $\mathcal{P}$ will complete absorbing gadgets in $\mathcal{A}$ to absorbers, while the remaining set of paths $\mathcal{P}' \subseteq \mathcal{P}$ will be used to connect all the absorbing gadgets in $\mathcal{A}$. More precisely, there is an enumeration $A_1, \dots, A_{|\mathcal{A}|}$ of $\mathcal{A}$ and an enumeration $P_1, \dots, P_{|\mathcal{A}|-1}$ of $\mathcal{P}'$ such that each P_i joins A_i to A_{i+1}. In particular, for each $i \in [|\mathcal{A}|] \setminus \{1, |\mathcal{A}|\}$, each vertex in $A_i \setminus V'$ is the endpoint of precisely one path in $\mathcal{P}$ (and thus has degree three in $\bigcup_{A \in \mathcal{A}} A \cup \bigcup_{P \in \mathcal{P}} P$), while both $A_1 \setminus V'$ and $A_{|\mathcal{A}|} \setminus V'$ contain precisely one vertex which is not the endpoint of some path in $\mathcal{P}$ (and thus these two vertices have degree two in $\bigcup_{A \in \mathcal{A}} A \cup \bigcup_{P \in \mathcal{P}} P$). Any tail T of an H-absorber $(\mathcal{A}, \mathcal{P})$ has to start at one of these two vertices. Altogether this means that, given any matching M of H, the path absorbing M in $(\mathcal{A}, \mathcal{P}, T)$ in the definition above actually exists.

Our absorbing structure is essentially an H-absorber $(\mathcal{A}, \mathcal{P})$ with a tail T and flexible sets $V_{\mathrm{flex}}, C_{\mathrm{flex}} \subseteq V(H)$ for an appropriately chosen template H. If $H - (X \cup Y)$ has a perfect matching M, then the path absorbing M in $(\mathcal{A}, \mathcal{P}, T)$ satisfies (†). This fact motivates the property of H that we need in the next definition.

First, we briefly discuss the purpose of the tail T. Recall that we reserve $\Theta(\mu n)$ vertices and colours for our absorbing structure. Our H-absorber, however, contains $\Theta(\eta n)$ vertices and colours, yet we need our absorbing structure to contain all but at most γn of the reserved vertices and colours. We need $\eta \ll \mu$ in order to build the H-absorber greedily, i.e. by selecting (v, c)-absorbing gadgets arbitrarily in turn for each edge of H, so the purpose of the tail then is to extend the H-absorber to a structure containing almost all of the reserved vertices and colours.

Definition 3.3. Let H be a bipartite graph with bipartition (A, B) such that $|A| = |B|$, and let $A' \subseteq A$ and $B' \subseteq B$ such that $|A'| = |B'|$.
  • We say H is *robustly matchable* with respect to A' and B' if for every pair of sets X and Y where $X \subseteq A'$, $Y \subseteq B'$, and $|X| = |Y| \le |A'|/2$, there is a perfect matching in $H - (X \cup Y)$.
  • In this case, we say A' and B' are *flexible* and $A \setminus A'$ and $B \setminus B'$ are *buffer sets*.

This concept was first introduced by Montgomery [31]. If H is robustly matchable with respect to V_{flex} and C_{flex}, then an H-absorber $(\mathcal{A}, \mathcal{P})$ with tail T satisfies (†). The last property of our absorbing structure that we need is that the flexible sets allow us to execute the covering step, which we capture in the following definition.

Definition 3.4. Let $V_{\mathrm{flex}} \subseteq V$, let $C_{\mathrm{flex}} \subseteq C$, and let G_{flex} be a spanning subgraph of K_n.
  • If $u, v \in V$ and $c \in C$, and $P \subseteq G_{\mathrm{flex}}$ is a rainbow path of length four such that
    – u and v are the ends of P,
    – $u', w, v' \in V_{\mathrm{flex}}$, where $uu', u'w, wv', vv' \in E(P)$,
    – $\phi(uu'), \phi(wv'), \phi(vv') \in C_{\mathrm{flex}}$, and

$- \phi(u'w) = c$,

then P is a $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$-*cover* of u, v, and c.

- If P is a rainbow path such that $P = \bigcup_{i=1}^{k} P_i$ where P_i is a $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$-cover of v_i, v_{i+1}, and c_i, then P is a $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$-*cover* of $\{v_1, \ldots, v_{k+1}\}$ and $\{c_1, \ldots, c_k\}$.
- If H is a regular bipartite graph with bipartition (V', C') where $V' \subseteq V$ and $C' \subseteq C$ such that
    - H is robustly matchable with respect to V_{flex} and C_{flex} where $|V_{\text{flex}}|, |C_{\text{flex}}| \geq \delta n$, and
    - for every $u, v \in V$ and $c \in C$, there are at least δn^2 $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$-covers of u, v, and c,

    then H is a δ-*absorbing template with flexible sets* $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$.
- If $(\mathcal{A}, \mathcal{P})$ is an H-absorber where H is a δ-absorbing template and T is a tail for $(\mathcal{A}, \mathcal{P})$, then $(\mathcal{A}, \mathcal{P}, T, H)$ is a δ-*absorber*.

A 36γ-absorber has the properties we need to execute both the covering step and the absorbing step, which we make formal with the next proposition.

First, we introduce the following convenient convention. Given $V'' \subseteq V' \subseteq V$ and $C'' \subseteq C' \subseteq C$, we say that (V'', C'') is *contained* in (V', C') with δ-*bounded remainder* if $V'' \subseteq V'$, $C'' \subseteq C'$, and $|V' \setminus V''|, |C' \setminus C''| \leq \delta n$. If G is a spanning subgraph of K_n, $V' \subseteq V$, and $C' \subseteq C$, then we say a graph G' is *contained* in (V', C', G) with δ-*bounded remainder* if $(V(G'), \phi(G'))$ is contained in (V', C') with δ-bounded remainder and $G' \subseteq G$.

Proposition 3.5. *If $(\mathcal{A}, \mathcal{P}, T, H)$ is a δ-absorber and P' is a rainbow path contained in $(V \setminus V', C \setminus C', G')$ with $\delta/18$-bounded remainder where*
- $V' = \bigcup_{A \in \mathcal{A}} V(A) \cup \bigcup_{P \in \mathcal{P}} V(P) \cup V(T)$
- $C' = \bigcup_{A \in \mathcal{A}} \phi(A) \cup \bigcup_{P \in \mathcal{P}} \phi(P) \cup \phi(T)$*, and*
- G' *is the complement of* $\bigcup_{A \in \mathcal{A}} A \cup \bigcup_{P \in \mathcal{P}} P \cup T$,

then there is both a rainbow Hamilton path containing P' and a rainbow cycle containing P' and all of the colours in C.

Proof. Order the colours in $C \setminus (\phi(P') \cup C')$ as $c_1, \ldots, c_k$, and note that $k \leq \delta n/18$. Order the vertices in $V \setminus (P' \cup V')$ as $v_1, \ldots, v_\ell$. Using that H is regular, it is easy to see that $|V'| = |C'| + 1$, and thus $\ell = k - 1$. Let v_0 and v_0' be the ends of P', let u be the end of T not in $V(A)$ for any $A \in \mathcal{A}$, and let u' be the unique vertex in $\bigcup_{A \in \mathcal{A}} V(A) \setminus V(T)$ of degree two in $\bigcup_{A \in \mathcal{A}} A \cup \bigcup_{P \in \mathcal{P}} P$. Let $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$ be the flexible sets of H.

First we show that there is a rainbow Hamilton path containing P'. We claim there is a $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$-cover P'' of $\{v_0, \ldots, v_k\}$ and $\{c_1, \ldots, c_k\}$, where $v_k := u$. Suppose for $j \in [k-1]$ and $i < j$ that P_i is a $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$-cover of v_i, v_{i+1}, and c_{i+1} such that $\bigcup_{i<j} P_i$ is a rainbow path. We show that there exists a $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$-cover P_j of v_j, v_{j+1}, and c_{j+1} that is internally-vertex- and colour-disjoint from $\bigcup_{i<j} P_i$, which implies that $\bigcup_{i\leq j} P_i$ is a rainbow path, and thus we can choose the path P'' greedily, proving the claim. Since each vertex in V_{flex} and each colour in C_{flex} is contained in at most $3n$ $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$-covers of v_j, v_{j+1}, and c_{j+1}, and since H is a δ-absorbing template, there are at least $\delta n^2 - 18n \cdot j$ $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$-covers of v_j, v_{j+1}, and c_{j+1} not containing a vertex or colour from $\bigcup_{i<j} P_i$. Thus, since $j < k \leq \delta n/18$, there exists a $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$-cover P_j of v_j, v_{j+1}, and c_{j+1} such that $\bigcup_{i\leq j} P_i$ is a rainbow path, as desired, and consequently we can choose the path P'' greedily, as claimed.

Now let $X := V(P'') \cap V_{\text{flex}}$, and let $Y := \phi(P'') \cap C_{\text{flex}}$. Since $|X| = |Y| = 3k \leq |V_{\text{flex}}|/2$ and H is robustly matchable with respect to V_{flex} and C_{flex}, there is a perfect matching M in $H - (X \cup Y)$. Let P''' be the path absorbing M in $(\mathcal{A}, \mathcal{P}, T)$. Then $P' \cup P'' \cup P'''$ is a rainbow Hamilton path, as desired.

Now we show that there is a rainbow cycle containing P' and all of the colours in C. By the same argument as before, there is a $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$-cover P_1'' of $\{v_0, \ldots, v_{\ell-1}, u\}$ and $\{c_1, \ldots, c_{k-1}\}$ as well as a $(V_{\text{flex}}, C_{\text{flex}}, G_{\text{flex}})$-cover P_2'' of v_0', u', and c_k such that P_1'' and P_2'' are vertex-and colour-disjoint. Letting $X := V(P_1'' \cup P_2'') \cap V_{\text{flex}}$ and $Y := \phi(P_1'' \cup P_2'') \cap C_{\text{flex}}$,

letting M be a perfect matching in $H - (X \cup Y)$ and P''' be the path absorbing M in $(\mathcal{A}, \mathcal{P}, T)$ as before, $P' \cup P_1'' \cup P_2'' \cup P'''$ is a rainbow cycle using all the colours in C, as desired.   □

3.2. **The proof of Theorem 1.3 when n is even.** In this subsection, we prove the n even case of Theorem 1.3 subject to two lemmas, Lemmas 3.8 and 3.9, which we prove in Sections 6 and 5, respectively. The first of these lemmas, Lemma 3.8, states that almost all 1-factorizations have two key properties, introduced in the next two definitions. Lemma 3.9 states that if a 1-factorization has both of these properties, then we can build an absorber using the reserved vertices and colours with high probability.

Recall the hierarchy of constants $\varepsilon, \gamma, \eta, \mu$ from (3.1). Firstly, we will need to show that if $G \in \mathcal{G}_{[n-1]}^{\mathrm{col}}$ is chosen uniformly at random, then with high probability, for any $V' \subseteq V$, $C' \subseteq C$ that are not too small, G admits many edges with colour in C' and both endpoints in V'. This property will be used in the construction of the tail of our absorber.

Definition 3.6. For $D \subseteq C = [n-1]$, we say that $G \in \mathcal{G}_D^{\mathrm{col}}$ is ε-*locally edge-resilient* if for all sets of colours $D' \subseteq D$ and all sets of vertices $V' \subseteq V$ of sizes $|V'|, |D'| \geq \varepsilon n$, we have that $e_{V', D'}(G) \geq \varepsilon^3 n^2 / 100$.

Secondly, we will need that almost all $G \in \mathcal{G}_{[n-1]}^{\mathrm{col}}$ contain many (v, c)-absorbing gadgets for all $v \in V$, $c \in C$.

Definition 3.7. Let $D \subseteq C = [n-1]$.
  - For $G \in \mathcal{G}_D^{\mathrm{col}}$, $x \in V$, $c \in D$, and $t \in \mathbb{N}_0$, we say that a collection $\mathcal{A}_{(x,c)}$ of (x, c)-absorbing gadgets in G is t-*well-spread* if
    - for all $v \in V$, there are at most t (x, c)-absorbing gadgets in $\mathcal{A}_{(x,c)}$ using v;
    - for all $e \in E(G)$, there are at most t (x, c)-absorbing gadgets in $\mathcal{A}_{(x,c)}$ using e;
    - for all $d \in D$, there are at most t (x, c)-absorbing gadgets in $\mathcal{A}_{(x,c)}$ using d.
  - We say that $G \in \mathcal{G}_{[n-1]}^{\mathrm{col}}$ is μ-*robustly gadget-resilient* if for all $x \in V$ and all $c \in C$, there is a $5\mu n/4$-well-spread collection of at least $\mu^4 n^2 / 2^{23}$ (x, c)-absorbing gadgets in G.

Lemma 3.8. *Suppose $1/n \ll \varepsilon, \mu \ll 1$. If ϕ is a 1-factorization of K_n chosen uniformly at random, then ϕ is ε-locally edge-resilient and μ-robustly gadget-resilient with high probability.*

We prove Lemma 3.8 using a 'coloured version' of switching arguments that are commonly used to study random regular graphs. Unfortunately, 1-factorizations of the complete graph K_n are 'rigid' structures, in the sense that it is difficult to make local changes without global ramifications on such a 1-factorization. Thus, instead of analysing switchings between graphs in $\mathcal{G}_{[n-1]}^{\mathrm{col}}$, we will analyse switchings between graphs in $\mathcal{G}_D^{\mathrm{col}}$ for appropriately chosen $D \subsetneq [n-1]$. In the setting of random Latin squares, this approach was used by McKay and Wanless [30] and further developed by Kwan and Sudakov [27], and we build on their ideas. Finally, we use results on the number of 1-factorizations of dense regular graphs due to Kahn and Lovász (see Theorem 6.10) and Ferber, Jain, and Sudakov (see Theorem 6.11) to study the number of completions of a graph $H \in \mathcal{G}_D^{\mathrm{col}}$ to a graph $G \in \mathcal{G}_{[n-1]}^{\mathrm{col}}$, and we use this information to compare the probability space corresponding to a uniform random choice of $H \in \mathcal{G}_D^{\mathrm{col}}$, with the probability space corresponding to a uniform random choice of $G \in \mathcal{G}_{[n-1]}^{\mathrm{col}}$.

Lemma 3.9. *Suppose $1/n \ll \varepsilon \ll \gamma \ll \eta \ll \mu \ll 1$, and let $p = q = \beta = 5\mu + 26887\eta/2 + \gamma/3 - 26880\varepsilon$. If ϕ is an ε-locally edge-resilient and μ-robustly gadget-resilient 1-factorization of K_n with vertex set V and colour set C and*
  (R1) V' is a p-random subset of V,
  (R2) C' is a q-random subset of C, and
  (R3) G' is a β-random subgraph of K_n,
then with high probability there is a 36γ-absorber $(\mathcal{A}, \mathcal{P}, T, H)$ such that $\bigcup_{A \in \mathcal{A}} A \cup \bigcup_{P \in \mathcal{P}} P \cup T$ is contained in (V', C', G') with γ-bounded remainder.

The final ingredient in the proof of Theorem 1.3 is the following lemma which follows from [16, Lemma 16], that enables us to find the long rainbow path whose leftover we absorb using the absorber from Lemma 3.9.

Lemma 3.10. *Suppose $1/n \ll \gamma \ll p$, and let $q = \beta = p$. If ϕ is a 1-factorization of K_n with vertex set V and colour set C and*
- *V' is a p-random subset of V,*
- *C' is a q-random subset of C, and*
- *G is a β-random subgraph of K_n,*

then with high probability there is a rainbow path contained in (V', C', G) with γ-bounded remainder.

We conclude this section with a proof of Theorem 1.3 in the case that n is even, assuming Lemmas 3.8 and 3.9.

Proof of Theorem 1.3, n even case. By Lemma 3.8, it suffices to prove that if ϕ is an ε-locally edge-resilient and μ-robustly gadget-resilient 1-factorization, then there is a rainbow Hamilton path and a rainbow cycle containing all of the colours.

Let $p = q = \beta$ as in Lemma 3.9, let V_1, V_2 be a random partition of V where V_1 is p-random and V_2 is $(1-p)$-random, let C_1, C_2 be a random partition of C where C_1 is q-random and C_2 is $(1-q)$-random, and let G_1 and G_2 be β-random and $(1-\beta)$-random subgraphs of K_n such that $E(G_1)$ and $E(G_2)$ partition the edges of K_n. By Lemma 3.9 applied with $V' = V_1$, $C' = C_1$, and $G' = G_1$, and by Lemma 3.10 applied with $V' = V_2$, $C' = C_2$, and $G = G_2$, the following holds with high probability. There exists

(i) a 36γ-absorber $(\mathcal{A}, \mathcal{P}, T, H)$ such that $\bigcup_{A \in \mathcal{A}} A \cup \bigcup_{P \in \mathcal{P}} P \cup T$ is contained in (V_1, C_1, G_1) with γ-bounded remainder, and

(ii) a rainbow path P' contained in (V_2, C_2, G_2) with γ-bounded remainder.

Now we fix an outcome of the random partitions (V_1, V_2), (C_1, C_2), and (G_1, G_2) so that (i) and (ii) hold. By Proposition 3.5, there is both a rainbow Hamilton path containing P' and a rainbow cycle containing P' and all of the colours in C, as desired. $\square$

4. Tools

In this section, we collect some results that we will use throughout the paper.

4.1. Probabilistic tools.
We will use the following standard probabilistic estimates.

Lemma 4.1 (Chernoff Bound). *Let X have binomial distribution with parameters n, p. Then for any $0 < t \leq np$,*

$$\mathbb{P}\left[|X - np| > t\right] \leq 2\exp\left(\frac{-t^2}{3np}\right).$$

Let $X_1, \ldots, X_m$ be independent random variables taking values in $\mathcal{X}$, and let $f : \mathcal{X}^m \to \mathbb{R}$. If for all $i \in [m]$ and $x_i', x_1, \ldots, x_m \in \mathcal{X}$, we have

$$|f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) - f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_m)| \leq c_i,$$

then we say X_i *affects* f by at most c_i.

Theorem 4.2 (McDiarmid's Inequality). *If $X_1, \ldots, X_m$ are independent random variables taking values in $\mathcal{X}$ and $f : \mathcal{X}^m \to \mathbb{R}$ is such that X_i affects f by at most c_i for all $i \in [m]$, then for all $t > 0$,*

$$\mathbb{P}\left[|f(X_1, \ldots, X_m) - \mathbb{E}[f(X_1, \ldots, X_m)]| \geq t\right] \leq \exp\left(-\frac{t^2}{\sum_{i=1}^m c_i^2}\right).$$

4.2. **Hypergraph matchings.** When we build our absorber in the proof of Lemma 3.9, we seek to efficiently use the vertices, colours, and edges of our random subsets $V' \subseteq V$, $C' \subseteq C$, $E' \subseteq E$, and to do this we make use of the existence of large matchings in almost-regular hypergraphs with small codegree. In fact, we will need the stronger property that there exists a large matching in such a hypergraph which is well-distributed with respect to a specified collection of vertex subsets. We make this precise in the following definition. Given a hypergraph $\mathcal{H}$ and a collection of subsets $\mathcal{F}$ of $V(\mathcal{H})$, we say a matching $\mathcal{M}$ in $\mathcal{H}$ is $(\gamma, \mathcal{F})$-*perfect* if for each $F \in \mathcal{F}$, at most $\gamma \cdot \max\{|F|, |V(\mathcal{H})|^{2/5}\}$ vertices of $\mathcal{F}$ are left uncovered by $\mathcal{M}$. The following theorem is a consequence of Theorem 1.2 in [4], and is based on a result of Pippenger and Spencer [33].

Theorem 4.3. *Suppose $1/n \ll \varepsilon \ll \gamma \ll 1/r$. Let $\mathcal{H}$ be an r-uniform hypergraph on n vertices such that for some $D \in \mathbb{N}$, we have $d_{\mathcal{H}}(x) = (1 \pm \varepsilon)D$ for all $x \in V(\mathcal{H})$ and $\Delta^c(\mathcal{H}) \leq D/\log^{9r} n$. If $\mathcal{F}$ is a collection of subsets of $V(\mathcal{H})$ such that $|\mathcal{F}| \leq n^{\log n}$, then there exists a $(\gamma, \mathcal{F})$-perfect matching.*

We will use Theorem 4.3 in the final step of constructing an absorber (see Lemma 5.5). We construct an auxiliary hypergraph $\mathcal{H}$ whose edges represent structures we wish to find, and a large well-distributed matching in $\mathcal{H}$ corresponds to an efficient allocation of vertices, colours, and edges of the 1-factorization to construct almost all of these desired structures. We remark that this is also a key strategy in the proof of Lemma 3.10, and was first used in [25].

4.3. **Robustly matchable bipartite graphs of constant degree.** In this subsection, we prove that there exist large bipartite graphs which are robustly matchable as in Definition 3.3, and have constant maximum degree.

Definition 4.4. Let $m \in \mathbb{N}$.
- An $RMBG(3m, 2m, 2m)$ is a bipartite graph H with bipartition $(A, B_1 \cup B_2)$ where $|A| = 3m$ and $|B_1| = |B_2| = 2m$ such that for any $B' \subseteq B_1$ of size m, there is a perfect matching in $H - B'$. In this case, we say H is *robustly matchable* with respect to B_1, and that B_1 is the identified *flexible set*.
- A $2RMBG(7m, 2m)$ is a bipartite graph H with bipartition (A, B) where $|A| = |B| = 7m$ such that H is robustly matchable with respect to sets $A' \subseteq A$ and $B' \subseteq B$ where $|A'| = |B'| = 2m$.

By [31, Lemma 10.7], for all sufficiently large m there exists an $RMBG(3m, 2m, 2m)$ with maximum degree at most 100. We use a one-sided (there is one flexible set) $RMBG(3m, 2m, 2m)$ exhibited in [16, Corollary 10] in which each of the vertex classes are regular, to construct a 256-regular two-sided (in that we identify a flexible set on each side of the vertex bipartition) $2RMBG(7m, 2m)$.

Lemma 4.5. *For all sufficiently large m, there is a $2RMBG(7m, 2m)$ that is 256-regular.*

Proof. Suppose that $m \in \mathbb{N}$ is sufficiently large. By [16, Corollary 10], there exists an $RMBG(3m, 2m, 2m)$ that is $(256, 192)$-regular (i.e. all vertices in the first vertex class have degree 256 and all vertices in the second vertex class of have degree 192). Let H and H' be two vertex-disjoint isomorphic copies of a $(192, 256)$-regular $RMBG(3m, 2m, 2m)$, and let $(A, B_1 \cup B_2)$ and $(A', B_1' \cup B_2')$ be the bipartitions of H and H' respectively such that H is robustly matchable with respect to B_1 and H' is robustly matchable with respect to B_1'.

Let H'' be a 64-regular bipartite graph with bipartition $(B_1 \cup B_2, B_1' \cup B_2')$ such that $H''[B_1 \cup B_1']$ contains a perfect matching M. We claim that $H \cup H' \cup H''$ is robustly matchable with respect to B_1 and B_1'. To that end, let $X \subseteq B_1$ and $Y \subseteq B_1'$ such that $|X| = |Y| \leq m$. It suffices to show that $H \cup H' \cup H'' - (X \cup Y)$ has a perfect matching. Since $H''[B_1 \cup B_1']$ contains a perfect matching, $H''[B_1 \cup B_1'] - (X \cup Y)$ contains a matching of size at least $2m - |X| - |Y| = 2(m - |X|)$. Thus, there exists a matching M' in $H''[B_1 \cup B_1'] - (X \cup Y)$ of size $m - |X|$. Let $X' := X \cup (B_1 \cap V(M'))$ and $Y' := Y \cup (B_1' \cap V(M'))$, and note that $|X'| = |Y'| = m$. Since H is an $RMBG(3m, 2m, 2m)$, $H - X'$ has a perfect matching M_1, and similarly $H' - Y'$ has a perfect matching M_2. Now $M' \cup M_1 \cup M_2$ is a perfect matching in $H \cup H' \cup H'' - (X \cup Y)$, as required. Since $H \cup H' \cup H''$ is 256-regular, the result follows. $\square$

5. Constructing the absorber: proof of Lemma 3.9

Throughout this section, let ϕ be an ε-locally edge-resilient and μ-robustly gadget resilient 1-factorization of K_n with vertex set V and colour set C, let $E := E(K_n)$, and recall

$$1/n \ll \varepsilon \ll \gamma \ll \eta \ll \mu \ll 1.$$

Let $\tilde{H}$ be a 256-regular $2RMBG(7m, 2m)$ where $2m = (\eta - 2\varepsilon)n$, which exists by Lemma 4.5. We define the following probabilities:

(5.1)
$$
\begin{aligned}
p_{\mathrm{flex}} &:= \eta, & q_{\mathrm{flex}} &:= \eta, \\
p_{\mathrm{buff}} &:= 5\eta/2, & q_{\mathrm{buff}} &:= 5\eta/2, \\
p_{\mathrm{abs}} &:= 6|E(\tilde{H})|/n + 2\mu, & q_{\mathrm{abs}} &:= 3|E(\tilde{H})|/n + \mu, \\
p_{\mathrm{link}} &:= 9|E(\tilde{H})|/n + 3\mu, & q_{\mathrm{link}} &:= 12|E(\tilde{H})|/n + 4\mu, \\
p'_{\mathrm{link}} &:= \gamma/3, & q'_{\mathrm{link}} &:= \gamma/3,
\end{aligned}
$$

and we let $p_{\mathrm{main}} := 1 - p_{\mathrm{flex}} - p_{\mathrm{buff}} - p_{\mathrm{abs}} - p_{\mathrm{link}} - p'_{\mathrm{link}}$ and $q_{\mathrm{main}} := 1 - q_{\mathrm{flex}} - q_{\mathrm{buff}} - q_{\mathrm{abs}} - q_{\mathrm{link}} - q'_{\mathrm{link}}$. Note that $p_{\mathrm{main}} = q_{\mathrm{main}}$, and let $\beta := 1 - p_{\mathrm{main}}$.

Definition 5.1. An *absorber partition of V, C, and K_n* is defined as follows:

(5.2)
$$
\begin{aligned}
V &= V_{\mathrm{main}} \mathbin{\dot\cup} V_{\mathrm{flex}} \mathbin{\dot\cup} V_{\mathrm{buff}} \mathbin{\dot\cup} V_{\mathrm{abs}} \mathbin{\dot\cup} V_{\mathrm{link}} \mathbin{\dot\cup} V'_{\mathrm{link}}, \text{ and} \\
C &= C_{\mathrm{main}} \mathbin{\dot\cup} C_{\mathrm{flex}} \mathbin{\dot\cup} C_{\mathrm{buff}} \mathbin{\dot\cup} C_{\mathrm{abs}} \mathbin{\dot\cup} C_{\mathrm{link}} \mathbin{\dot\cup} C'_{\mathrm{link}},
\end{aligned}
$$

where V_{main} is p_{main}-random, V_{flex} is p_{flex}-random etc, and the sets of colours are defined analogously. Let $V' := V \setminus V_{\mathrm{main}}$, $C' := C \setminus C_{\mathrm{main}}$, and let G' be a β-random subgraph of K_n.

Note that V', C', and G' satisfy (R1)–(R3) in the statement of Lemma 3.9.

5.1. Overview of the proof.
We now overview our strategy for proving Lemma 3.9. First we need the following definitions. A *link* is a rainbow path of length 4 with internal vertices in $V_{\mathrm{link}} \cup V'_{\mathrm{link}}$, ends in V_{abs}, and colours and edges in $C_{\mathrm{link}} \cup C'_{\mathrm{link}}$ and G', respectively. A link with internal vertices in V_{link} and colours in C_{link} is a *main link*, and a link with internal vertices in V'_{link} and colours in C'_{link} is a *reserve link*. If M is a matching and $\mathcal{P} = \{P_e\}_{e \in E(M)}$ is a collection of vertex-disjoint links such that $\bigcup_{P \in \mathcal{P}} P$ is rainbow and P_{uv} has ends u and v for every $uv \in E(M)$, then $\mathcal{P}$ *links M*.

We aim to build a 36γ-absorber $(\mathcal{A}, \mathcal{P}, T, H)$ such that $\bigcup_{A \in \mathcal{A}} A \cup \bigcup_{P \in \mathcal{P}} P \cup T$ is contained in (V', C', G') with γ-bounded remainder and $H \cong \tilde{H}$. First, we show (see Lemma 5.3) that with high probability there is a 36γ-absorbing template $H \cong \tilde{H}$, where

- H has flexible sets $(V'_{\mathrm{flex}}, C'_{\mathrm{flex}}, G')$ and $(V'_{\mathrm{flex}}, C'_{\mathrm{flex}})$ is contained in $(V_{\mathrm{flex}}, C_{\mathrm{flex}})$ with 3ε-bounded remainder, and
- H has buffer sets V'_{buff} and C'_{buff} where $(V'_{\mathrm{buff}}, C'_{\mathrm{buff}})$ is contained in $(V_{\mathrm{buff}}, C_{\mathrm{buff}})$ with 6ε-bounded remainder.

Then, we show that with high probability, there exists an H-absorber $(\mathcal{A}, \mathcal{P})$ where

- for every $vc \in E(H)$, the (v,c)-absorbing gadget $A_{v,c} \in \mathcal{A}$ uses vertices, colours, and edges in V_{abs}, C_{abs}, and G', respectively, and
- every $P \in \mathcal{P}$ is a link.

In particular, if $\mathcal{A} = \{A_1, \ldots, A_k\}$, where A_i is a (v_i, c_i)-absorbing gadget, then $\mathcal{P}$ links the matching $M_1 \cup M_2 \cup M_3$, where $V(M_1)$, $V(M_2)$, and $V(M_3)$ are pairwise vertex-disjoint, and

(M1) $M_1 = \{r_1 s_1, \ldots, r_k s_k\}$, where r_i and s_i are non-adjacent vertices of the 4-cycle in A_i, for each $i \in [k]$,

(M2) $M_2 = \{w_1 x_1, \ldots, w_k x_k\}$, where w_i is a non-v_i vertex of the triangle in A_i and x_i is a vertex of the 4-cycle in A_i, for each $i \in [k]$, and

(M3) $M_3 = \{y_1 z_2, \ldots, y_{k-1} z_k\}$, where y_i is a non-v_i vertex of the triangle in A_i for each $i \in [k-1]$, and z_i is a vertex of the 4-cycle in A_i for each $i \in [k] \setminus \{1\}$.

Finally, letting V'_{abs} and C'_{abs} be the vertices and colours in V_{abs} and C_{abs} not used by any (v,c)-absorbing gadget in $\mathcal{A}$, we show that with high probability there is a tail T for $(\mathcal{A}, \mathcal{P})$ where T is the union of
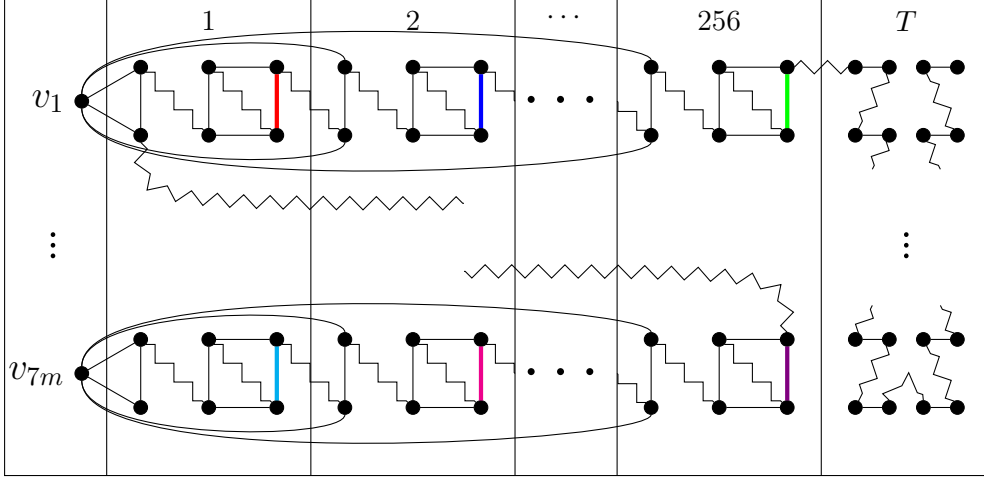
FIGURE 3. An absorber $(\mathcal{A}, \mathcal{P}, T, H)$, where $\mathcal{P}$ links $\bigcup_{i=1}^{3} M_i$ and $T = M \cup \bigcup_{P \in \mathcal{T}} P$, where $\mathcal{T}$ links M_4. Links are drawn as zigzags.

- a rainbow matching M contained in $(V'_{\mathrm{abs}}, C'_{\mathrm{abs}}, G')$ with 6ε-bounded remainder and
- a collection $\mathcal{T}$ of vertex-disjoint links where all but one vertex in $V(M)$ is the end of precisely one link.

In particular, if $E(M) = \{a_1 b_1, \ldots, a_\ell b_\ell\}$, then $\mathcal{T}$ links M_4, where

(M4) M_4 is a matching of size ℓ with edges $b_i a_{i+1}$ for every $i \in [\ell - 1]$ and an edge va_1 where v is one of the two vertices used by a gadget in $\mathcal{A}$ that is not in a link in $\mathcal{P}$.

See Figure 3.

Fact 5.2. *Suppose that $\mathcal{A}$ satisfies H. If $\mathcal{P} \cup \mathcal{T}$ links $M_1 \cup \cdots \cup M_4$, where $\mathcal{P}$ links $M_1 \cup M_2 \cup M_3$ and $\mathcal{T}$ links M_4, then $\mathcal{P}$ completes $\mathcal{A}$ and thus $(\mathcal{A}, \mathcal{P})$ is an H-absorber. Moreover, $T := M \cup \bigcup_{P \in \mathcal{T}} P$ is a tail of $(\mathcal{A}, \mathcal{P})$. Thus $(\mathcal{A}, \mathcal{P}, T, H)$ is a 36γ-absorber.*

We find these structures in the following steps. For Steps 1 and 2, see Lemma 5.4, and for Steps 3 and 4, see Lemma 5.5.

1) First, we find the collection $\mathcal{A}$ of absorbing gadgets greedily, using the robust gadget-resilience property of ϕ,
2) then we greedily construct the matching M, using the local edge-resilience property of ϕ.
3) Next, we construct an auxiliary hypergraph in which each hyperedge corresponds to a main link and apply Theorem 4.3 to choose most of the links in $\mathcal{P}$, and
4) finally we greedily choose the remainder of the links in $\mathcal{P}$ from the reserve links.

5.2. The absorbing template.

Lemma 5.3. *Consider an absorber partition of V, C, and K_n. With high probability, there exists a 36γ-absorbing template $H \cong \tilde{H}$, where*

(5.3.1) *H has flexible sets $(V'_{\mathrm{flex}}, C'_{\mathrm{flex}}, G')$ where (V'_{flex}, C'_{flex}) is contained in $(V_{\mathrm{flex}}, C_{\mathrm{flex}})$ with 3ε-bounded remainder, and*

(5.3.2) *H has buffer sets V'_{buff} and C'_{buff} where $(V'_{\mathrm{buff}}, C'_{\mathrm{buff}})$ is contained in $(V_{\mathrm{buff}}, C_{\mathrm{buff}})$ with 6ε-bounded remainder.*

Proof. For convenience, let $p := p_{\mathrm{flex}}$ and $q := q_{\mathrm{flex}}$. We claim that the following holds with high probability:

(a) $|V_{\mathrm{flex}}|, |C_{\mathrm{flex}}| = (\eta \pm \varepsilon)n$,
(b) $|V_{\mathrm{buff}}|, |C_{\mathrm{buff}}| = (5\eta/2 \pm \varepsilon)n$, and
(c) for every distinct $u, v \in V$ and $c \in C$, there are at least $p^3 q^3 \beta^4 n^2 / 4$ $(V_{\mathrm{flex}}, C_{\mathrm{flex}}, G')$-covers of u, v, and c.

Indeed, (a) and (b) follow from the Chernoff Bound (Lemma 4.1). To prove (c), for each u, v, and c, we apply McDiarmid's Inequality (Theorem 4.2). Consider the random variable f counting the number of $(V_{\text{flex}}, C_{\text{flex}}, G')$-covers of u, v, and c. Note that f is determined by the following independent binomial random variables: $\{X_z\}_{z \in V}$, where X_z indicates if $z \in V_{\text{flex}}$, $\{X_{c'}\}_{c' \in C}$, where $X_{c'}$ indicates if $c' \in C_{\text{flex}}$, and for each edge e, the random variable X_e which indicates if $e \in E(G')$. We claim there are at least $2(n/2 - 2)(n - 7)$ (V, C, K_n)-covers of u, v, and c. To that end, let $u'w$ be a c-edge with $u', w \in V \setminus \{u, v\}$. There are at least $n - 7$ vertices $v' \in V \setminus \{u, v, u', w\}$ such that $\phi(vv'), \phi(wv') \notin \{\phi(uu'), c\}$, and for each such vertex v' the path $uu'wv'v$ is a (V, C, K_n)-cover of u, v, and c. Similarly, there are at least $n - 7$ (V, C, K_n)-covers of the form $uwu'v'v$. Altogether this gives at least $2(n/2 - 2)(n - 7) \geq n^2/2$ (V, C, K_n)-covers of u, v, and c, as claimed. Therefore $\mathbb{E}[f] \geq p^3 q^3 \beta^4 n^2/2$. For each $z \in V$, X_z affects f by at most $3n$, and X_{uz}, and X_{vz} each affect f by at most n, and for each $c' \in C$, $X_{c'}$ affects f by at most $3n$. For each edge e not incident to u or v, if e is a c-edge, then X_e affects f by at most $2n$, and otherwise e affects f by at most two. Thus, by McDiarmid's Inequality applied with $t = \mathbb{E}[f]/2$, there are at least $p^3 q^3 \beta^4 n^2/4$ $(V_{\text{flex}}, C_{\text{flex}}, G')$-covers of u, v, and c with probability at least $1 - \exp\left(-p^6 q^6 \beta^8 n^4/O(n^3)\right)$. Thus by a union bound, (c) also holds with high probability.

Now we assume (a)–(c) holds, and we show there exists a 36γ-absorbing template $H \cong \tilde{H}$ satisfying (5.3.1) and (5.3.2).

Since $m = (\eta/2 - \varepsilon)n$, by (a) and (b), there exists $V'_{\text{flex}} \subseteq V_{\text{flex}}$, $C'_{\text{flex}} \subseteq C_{\text{flex}}$, $V'_{\text{buff}} \subseteq V_{\text{buff}}$, and $C'_{\text{buff}} \subseteq C_{\text{buff}}$, such that $|V'_{\text{flex}}|, |C'_{\text{flex}}| = 2m$ and $|V'_{\text{buff}}|, |C'_{\text{buff}}| = 5m$, which we choose arbitrarily, and moreover, $|V_{\text{flex}} \setminus V'_{\text{flex}}|, |C_{\text{flex}} \setminus C'_{\text{flex}}| \leq 3\varepsilon n$ and $|V_{\text{buff}} \setminus V'_{\text{buff}}|, |C_{\text{buff}} \setminus C'_{\text{buff}}| \leq 6\varepsilon n$, as required. Choose bijections from $V'_{\text{flex}}, C'_{\text{flex}}$, V'_{buff}, and C'_{buff} to the flexible sets and the buffer sets of $\tilde{H}$ arbitrarily, and let $H \cong \tilde{H}$ be the corresponding graph. Now H satisfies (5.3.1) and (5.3.2), as required, so it remains to show that H is a 36γ-absorbing template. Since each vertex or colour in V_{flex} or C_{flex} is in at most $3n$ $(V_{\text{flex}}, C_{\text{flex}}, G')$-covers of u, v, and c, (a) and (c) imply that there are at least $p^3 q^3 \beta^4 n^2/4 - 18\varepsilon n^2 \geq 36\gamma n^2$ $(V'_{\text{flex}}, C'_{\text{flex}}, G')$-covers of u, v, and c, so H is a 36γ-absorbing template, as desired. $\qquad \square$

5.3. Greedily building an H-absorber.

Lemma 5.4. *Consider an absorber partition of V, C, and K_n. The following holds with high probability. Suppose $V_{res} \subseteq V_{flex} \cup V_{buff}$ and $C_{res} \subseteq C_{flex} \cup C_{buff}$. For every graph $H \cong \tilde{H}$ with bipartition $(V_{\text{res}}, C_{\text{res}})$, there exists*

(5.4.1) *a collection $\mathcal{A} = \{A_{vc} : vc \in E(H)\}$ such that $\mathcal{A}$ satisfies H and such that for all $A_{vc} \in \mathcal{A}$ we have that A_{vc} uses vertices, colours, and edges in V_{abs}, C_{abs}, and G' respectively, and*

(5.4.2) *a rainbow matching M contained in $(V'_{\text{abs}}, C'_{\text{abs}}, G')$ with 5ε-bounded remainder, where V'_{abs} and C'_{abs} are the sets of vertices and colours in V_{abs} and C_{abs} not used by any absorbing gadget in $\mathcal{A}$.*

Proof. For convenience, let $p := p_{\text{abs}}$ and $q := q_{\text{abs}}$ in this proof.

Since ϕ is μ-robustly gadget-resilient, for every $v \in V$, $c \in C$, there is a collection $\mathcal{A}_{v,c}$ of precisely $2^{-23} \mu^4 n^2$ (v, c)-absorbing gadgets such that every vertex, every colour, and every edge is used by at most $5\mu n/4$ of the $A \in \mathcal{A}_{v,c}$. (Recall from Definition 3.1 that a (v, c)-absorbing gadget does not use v and c.) Fix $v \in V$, $c \in C$. The expected number of the (v, c)-absorbing gadgets in $\mathcal{A}_{v,c}$ using only vertices in V_{abs}, colours in C_{abs}, and edges in G' is $p^6 q^3 \beta^7 |\mathcal{A}_{v,c}|$. Let $\mathcal{E}_{v,c}$ be the event that fewer than $p^6 q^3 \beta^7 |\mathcal{A}_{v,c}|/2$ of the (v, c)-absorbing gadgets in $\mathcal{A}_{v,c}$ use only vertices in V_{abs}, colours in C_{abs} and edges in G'. We claim that $\mathbb{P}[\mathcal{E}_{v,c}] \leq \exp(-2^{-51} p^{12} q^6 \beta^{14} \mu^6 n)$.

To see this, for each $u \in V$, $d \in C$, $e \in E$, let m_u, m_d, and m_e denote the number of (v, c)-absorbing gadgets in $\mathcal{A}_{v,c}$ using u, d, and e, respectively. We will apply McDiarmid's Inequality (Theorem 4.2) to the function $f_{v,c}$ which counts the number of $A \in \mathcal{A}_{v,c}$ using only vertices in V_{abs}, colours in C_{abs}, and edges in G'. We use independent indicator random variables $\{X_u\}_{u \in V} \cup \{X_d\}_{d \in C} \cup \{X_e\}_{e \in E}$ which indicate whether or not a vertex u is in V_{abs},

a colour d is in C_{abs}, and an edge e is in G'. Each random variable X_u, X_d, X_e affects $f_{v,c}$ by at most m_u, m_d, m_e, respectively. Since $m_u \leq 5\mu n/4$ for all $u \in V$ and $m_d \leq 5\mu n/4$ for all $d \in C$, we have $\sum_{u \in V} m_u^2, \sum_{d \in C} m_d^2 \leq 25\mu^2 n^3/16$. Since $\sum_{e \in E} m_e = 7|\mathcal{A}_{v,c}|$ and $m_e \leq 5\mu n/4$ for all $e \in E$, it follows that $\sum_{e \in E} m_e^2 \leq 35\mu n|\mathcal{A}_{v,c}|/4$. Therefore, by McDiarmid's Inequality, we have

$$\mathbb{P}\left[\mathcal{E}_{v,c}\right] \leq \exp\left(-\frac{p^{12}q^6\beta^{14}|\mathcal{A}_{v,c}|^2/4}{25\mu^2 n^3/8 + 35\mu n|\mathcal{A}_{v,c}|/4}\right) \leq \exp(-2^{-51}p^{12}q^6\beta^{14}\mu^6 n),$$

as claimed. Thus, by a union bound, the probability that there exist $v \in V$, $c \in C$ such that $\mathcal{E}_{v,c}$ holds is at most $\exp(-2^{-52}p^{12}q^6\beta^{14}\mu^6 n)$.

We claim the following holds with high probability:
(a) $|V_{\text{abs}}| = (p \pm \varepsilon)n$ and $|C_{\text{abs}}| = (q \pm \varepsilon)n$;
(b) for every $v \in V$, $c \in C$, the event $\mathcal{E}_{v,c}$ does not hold;
(c) for every $V^\circ \subseteq V_{\text{abs}}$ and $C^\circ \subseteq C_{\text{abs}}$ such that $|V^\circ|, |C^\circ| \geq \varepsilon n$, there are at least $\beta\varepsilon^3 n^2/200$ edges in G' with both ends in V° and a colour in C°.

Indeed, (a) holds by the Chernoff Bound (Lemma 4.1), we have already shown (b), and since ϕ is ε-locally edge-resilient, (c) holds by applying the Chernoff Bound for each V° and C° and using a union bound.

Now we assume that (a)–(c) hold, we suppose $H \cong \tilde{H}$ has bipartition $(V_{\text{res}}, C_{\text{res}})$ contained in $(V_{\text{flex}} \cup V_{\text{buff}}, C_{\text{flex}} \cup C_{\text{buff}})$, and we show that (5.4.1) and (5.4.2) hold. Arbitrarily order the edges of H as $e_1, \ldots, e_{|E(H)|}$. Let $i \in [|E(H)|]$ and suppose that for each $j < i$ we have found a (v_j, c_j)-absorbing gadget A_j, where $e_j = v_j c_j$, and further, the collection $\{A_1, \ldots, A_{i-1}\}$ satisfies the spanning subgraph of H containing precisely the edges $e_1, \ldots, e_{i-1}$. Writing $e_i = v_i c_i$, by (b) there is a collection $\mathcal{A}_{v_i,c_i}^{\text{abs}}$ of at least $2^{-24}p^6 q^3 \beta^7 \mu^4 n^2$ (v_i, c_i)-absorbing gadgets each using only V_{abs}-vertices, C_{abs}-colours, and G'-edges, and moreover, each vertex in V_{abs}, colour in C_{abs}, and edge in G' is used by at most $5\mu n/4$ of the $A \in \mathcal{A}_{v_i,c_i}^{\text{abs}}$. Thus, at most $20\mu n \cdot i \leq 20\mu n|E(H)| \leq 17920\eta\mu n^2$ of the (v_i, c_i)-absorbing gadgets in $\mathcal{A}_{v_i,c_i}^{\text{abs}}$ use a vertex, colour, or edge used by any of the A_j for $j < i$. Since $|\mathcal{A}_{v_i,c_i}^{\text{abs}}| \geq 2^{-24}p^6 q^3 \beta^7 \mu^4 n^2$, we conclude that there is at least one (v_i, c_i)-absorbing gadget $A \in \mathcal{A}_{v_i,c_i}^{\text{abs}}$ using vertices, colours, and edges which are disjoint from the vertices, colours, and edges used by A_j, for all $j < i$. We arbitrarily choose such an A to be A_i. Continuing in this way, it is clear that $\mathcal{A} := \{A_i\}_{i=1}^{|E(H)|}$ satisfies H, so (5.4.1) holds.

Now we prove (5.4.2). Let V_{abs}' and C_{abs}' be the vertices, colours, and edges in V_{abs} and C_{abs} not used by any (v, c)-absorbing gadget in $\mathcal{A}$. By (a) and (5.1), we have $|V_{\text{abs}}'| = (2\mu \pm \varepsilon)n$ and $|C_{\text{abs}}'| = (\mu \pm \varepsilon)n$. Thus, by (c), we can greedily choose a rainbow matching M in $(V_{\text{abs}}', C_{\text{abs}}', G')$ of size at least $(\mu - 2\varepsilon)n$, and M satisfies (5.4.2). □

5.4. **Linking.** Lastly, we need the following lemma, inspired by [16, Lemma 20], which we use to both complete the set of absorbing gadgets obtained by Lemma 5.4 to an H-absorber and also construct its tail. Recall that links were defined at the beginning of Section 5.1.

Lemma 5.5. *Consider an absorber partition of V, C, and K_n. The following holds with high probability. For every matching M such that $V(M) \subseteq V_{\text{abs}}$ and $|V_{\text{abs}} \setminus V(M)| \leq \varepsilon n$, there exists a collection $\mathcal{P}$ of links in G' such that*

(5.5.1) $\mathcal{P}$ links M and
(5.5.2) $\bigcup_{P \in \mathcal{P}} P \setminus V(M)$ is contained in $(V_{\text{link}} \cup V_{\text{link}}', C_{\text{link}} \cup C_{\text{link}}', G')$ with $\gamma/2$-bounded remainder.

Proof. We choose a new constant δ such that $\varepsilon \ll \delta \ll \gamma$. For convenience, let $p := p_{\text{link}}$ and $q := q_{\text{link}}$, let G_1 be the spanning subgraph of G' consisting of edges with a colour in C_{link}, and let G_2 be the spanning subgraph of G' consisting of edges with a colour in C_{link}'. First we claim that with high probability the following holds:
(a) $|V_{\text{link}}| = (p \pm \varepsilon)n$, $|C_{\text{link}}| = (q \pm \varepsilon)n$, $|V_{\text{link}}'| = (\gamma/3 \pm \varepsilon)n$, and $|C_{\text{link}}'| = (\gamma/3 \pm \varepsilon)n$,
(b) $|V_{\text{abs}}| = (1 \pm \varepsilon)p_{\text{abs}}n = (1 \pm \varepsilon)2pn/3$,
(c) for all $v \in V$, we have

      (i) $|N_{G_1}(v) \cap V_{\mathrm{abs}}| = (1 \pm \varepsilon)p_{\mathrm{abs}}\beta qn = (1 \pm \varepsilon)2p\beta qn/3$ and

      (ii) $|N_{G_1}(v) \cap V_{\mathrm{link}}| = (1 \pm \varepsilon)p\beta qn$,

  (d) for all $c \in C$, we have

      (i) $|E_{G'}^c(V_{\mathrm{abs}}, V_{\mathrm{link}})| = (1 \pm \varepsilon)p_{\mathrm{abs}}p\beta n = (1 \pm \varepsilon)2p^2\beta n/3$ and

      (ii) $|E^c(G'[V_{\mathrm{link}}])| = (1 \pm \varepsilon)p^2\beta n/2$,

  (e) for all distinct $u, v \in V$, we have $|N_{G_1}(u) \cap N_{G_1}(v) \cap V_{\mathrm{link}}| = (1 \pm \varepsilon)p\beta^2q^2n$, and

  (f) for all $u, v \in V$ we have $|N_{G_2}(u) \cap N_{G_2}(v) \cap V'_{\mathrm{link}}| \geq \gamma^6 n$.

Indeed (a)–(d) follow from (5.1) and the Chernoff Bound. We prove (e) and (f) using McDiarmid's Inequality. To prove (e), for each $u, v \in V$, we apply McDiarmid's Inequality to the random variable f counting $|N_{G_1}(u) \cap N_{G_1}(v) \cap V_{\mathrm{link}}|$ with respect to independent binomial random variables $\{X_w, X_{uw}, X_{vw}\}_{w \in V}$ and $\{X_c\}_{c \in C}$, where X_w indicates if $w \in V_{\mathrm{link}}$, X_{uw} and X_{vw} indicate if the edges uw and vw respectively are in G', and X_c indicates if $c \in C_{\mathrm{link}}$. For each $w \in V$, X_w, X_{uw}, and X_{vw} affect f by at most one, and for each $c \in C$, X_c affects f by at most two. Thus, by McDiarmid's Inequality with $t = \varepsilon\mathbb{E}\,[f]\,/2$, we have $|N_{G_1}(u) \cap N_{G_1}(v) \cap V_{\mathrm{link}}| = (1 \pm \varepsilon)p\beta^2q^2n$ with probability at least $1 - \exp\left(-(\varepsilon p\beta^2 q^2 n/2)^2/7n\right)$. By a union bound, (e) also holds with high probability. The proof of (f) is similar, so we omit it.

Now we assume (a)–(f) hold, we suppose M is a matching such that $V(M) \subseteq V_{\mathrm{abs}}$ and $|V_{\mathrm{abs}} \setminus V(M)| \leq \varepsilon n$, and we show that (5.5.1) and (5.5.2) hold with respect to M. Since $|V_{\mathrm{abs}} \setminus V(M)| \leq \varepsilon n$, (b) implies that

$$(5.3) \qquad\qquad |V(M)| = (1 \pm \sqrt{\varepsilon})2pn/3.$$

We apply Theorem 4.3 to the following 8-uniform hypergraph $\mathcal{H}$: the vertex-set is $E(M) \cup V_{\mathrm{link}} \cup C_{\mathrm{link}}$, and for every $xy \in E(M)$, $v_1, v_2, v_3 \in V_{\mathrm{link}}$, and $c_1, c_2, c_3, c_4 \in C_{\mathrm{link}}$, $\mathcal{H}$ contains the hyperedge $\{xy, v_1, v_2, v_3, c_1, c_2, c_3, c_4\}$ if there is a main link P such that

- P has ends x and y,
- v_1, v_2, and v_3 are the internal vertices in P, and
- $\phi(P) = \{c_1, c_2, c_3, c_4\}$.

Claim 1: $d_{\mathcal{H}}(v) = (1 \pm 2\sqrt{\varepsilon})p^3\beta^4q^4n^3$ for all $v \in V(\mathcal{H})$.

Proof of claim: Let $xy \in E(M)$. By (a), there are $(1 \pm \varepsilon)pn$ vertices $v_1 \in V_{\mathrm{link}}$ that can be in a link P with ends x and y corresponding to a hyperedge in $\mathcal{H}$, where v_1 is not adjacent to x or y. For each such $v_1 \in V_{\mathrm{link}}$, by (e), there are $(1 \pm \varepsilon)p\beta^2q^2n$ choices for the vertex in V_{link} adjacent to both x and v_1 in P, and for each such $v_2 \in V_{\mathrm{link}}$, again by (e), there are $(1 \pm 2\varepsilon)p\beta^2q^2n$ choices for the vertex in V_{link} adjacent to both v_1 and y in P such that P is a main link. Thus, $d_{\mathcal{H}}(xy) = (1 \pm 5\varepsilon)p^3\beta^4q^4n^3$, as required.

Now let $v_1 \in V_{\mathrm{link}}$. First, we count the number of hyperedges in $\mathcal{H}$ containing v_1 corresponding to a link P where v_1 is adjacent to one of the ends. By (c), and since $|V_{\mathrm{abs}} \setminus V(M)| \leq \varepsilon n$, there are $(1 \pm \sqrt{\varepsilon})2p\beta qn/3$ choices of the vertex $x \in V(M)$ adjacent to v_1 in P. For each such x, again by (c), there are $(1 \pm 2\varepsilon)p\beta qn$ choices of the vertex $v_2 \in V_{\mathrm{link}}$ adjacent to y in P where $xy \in E(M)$. For each such $v_2 \in V_{\mathrm{link}}$, by (e), there are $(1 \pm 2\varepsilon)p\beta^2q^2n$ choices of the vertex $v_3 \in V_{\mathrm{link}}$ adjacent to both v_1 and v_2 in P. Thus, the number of hyperedges in $\mathcal{H}$ containing v_1 corresponding to a link where v_1 is adjacent to one of the ends is $(1 \pm 2\sqrt{\varepsilon})2p^3\beta^4q^4n^3/3$.

Next, we count the number of hyperedges in $\mathcal{H}$ containing v_1 corresponding to a link P where v_1 is not adjacent to one of the ends. By (5.3), there are $(1 \pm \sqrt{\varepsilon})pn/3$ choices for the edge $xy \in E(M)$ where x and y are the ends of P. For each such $xy \in E(M)$, by (e), there are $(1 \pm \varepsilon)p\beta^2q^2n$ choices of the vertex $v_2 \in V_{\mathrm{link}}$ such that v_2 is adjacent to x and v_1 in P, and again by (e), for each such $v_2 \in V_{\mathrm{link}}$, there are $(1 \pm 2\varepsilon)p\beta^2q^2n$ choices of the vertex $v_3 \in V_{\mathrm{link}}$ adjacent to both y and v_1 in P. Thus, the number of hyperedges in $\mathcal{H}$ containing v_1 corresponding to a link where v_1 is not adjacent to one of the ends is $(1 \pm 2\sqrt{\varepsilon})p^3\beta^4q^4n^3/3$, so

$$d_{\mathcal{H}}(v_1) = (1 \pm 2\sqrt{\varepsilon})(2p^3\beta^4q^4n^3/3) + (1 \pm 2\sqrt{\varepsilon})p^3\beta^4q^4n^3/3 = (1 \pm 2\sqrt{\varepsilon})p^3\beta^4q^4n^3,$$

as required.

Now let $c_1 \in C_{\mathrm{link}}$. First we count the number of hyperedges in $\mathcal{H}$ containing c_1 corresponding to a link P where c_1 is the colour of one of the edges incident to an end of P. By (d), and since

$|V_{\text{abs}} \setminus V(M)| \leq \varepsilon n$, there are $(1 \pm \sqrt{\varepsilon})2p^2\beta n/3$ choices of the edge xv_1 in P where $x \in V(M)$ is an end of P and $\phi(xv_1) = c_1$. For each such edge xv_1, by (c), there are $(1 \pm 2\varepsilon)p\beta qn$ choices of the vertex $v_2 \in V_{\text{link}}$ adjacent to y in P where $xy \in E(M)$. For each such vertex v_2, by (e), there are $(1 \pm 2\varepsilon)p\beta^2q^2n$ choices of the vertex v_3 adjacent to both v_1 and v_2 in P. Thus, the number of hyperedges in $\mathcal{H}$ containing c_1 corresponding to a link where c_1 is the colour of one of the edges incident to an end of P is $(1 \pm 2\sqrt{\varepsilon})2p^4\beta^4q^3n^3/3$.

Next, we count the number of hyperedges in $\mathcal{H}$ containing c_1 corresponding to a link P where c_1 is the colour of one of the edges with both ends in V_{link}. By (d), there are $(1 \pm \varepsilon)p^2\beta n/2$ choices for the edge v_1v_2 in P such that $\phi(v_1v_2) = c_1$, and thus $(1 \pm \varepsilon)p^2\beta n$ choices for the edge if we assume v_1 is adjacent to an end in P. For each such edge v_1v_2, by (c), and since $|V_{\text{abs}} \setminus V(M)| \leq \varepsilon n$, there are $(1 \pm \sqrt{\varepsilon})2p\beta qn/3$ choices for the vertex $x \in V(M)$ adjacent to v_1 in P. For each such vertex x, by (e), there are $(1 \pm 2\varepsilon)p\beta^2q^2n$ choices for the vertex v_3 adjacent to both y and v_2 in P, where $xy \in E(M)$. Thus, the number of hyperedges in $\mathcal{H}$ containing c_1 corresponding to a link where c_1 is the colour of one of the edges with both ends in V_{link} is $(1 \pm 2\sqrt{\varepsilon})2p^4\beta^4q^3n^3/3$, so by (5.1)

$$d_{\mathcal{H}}(c_1) = (1 \pm 2\sqrt{\varepsilon})4p^4\beta^4q^3n^3/3 = (1 \pm 2\sqrt{\varepsilon})p^3\beta^4q^4n^3,$$

as required to prove Claim 1. ⊣

Claim 2: $\Delta^c(\mathcal{H}) \leq 100n^2$.

This can be proved similarly as above (with room to spare). Let $\mathcal{F} := \{E(M), V_{\text{link}}, C_{\text{link}}\}$. By Theorem 4.3, $\mathcal{H}$ has a $(\delta, \mathcal{F})$-perfect matching $\mathcal{M}$. Let $\mathcal{P}_1$ be the collection of links corresponding to $\mathcal{M}$, and let M' be the matching consisting of all those $xy \in E(M)$ that are not covered by $\mathcal{M}$. To complete the proof, we greedily find a collection $\mathcal{P}_2$ of reserve links that links M'.

Write $E(M') = \{x_1y_1, \ldots, x_ky_k\}$, and suppose P_i is a reserve link with ends x_i and y_i for $i < j$, where $j \in [k]$. We show that that there is a reserve link P_j that is vertex- and colour-disjoint from $\bigcup_{i<j} P_i$, which implies that $\bigcup_{i=1}^j P_i$ links $\{x_1y_1, \ldots, x_jy_j\}$, and thus we can choose $\mathcal{P}_2$ greedily. Since $k \leq \delta n$ and each link has at most three vertices in V'_{link}, by (a), there is a vertex $v \in V'_{\text{link}} \setminus \bigcup_{i<j} V(P_i)$. By (f), there are at least $\gamma^6 n - 11j$ vertices $v_1 \in (N_{G_2}(x_j) \cap N_{G_2}(v) \cap V'_{\text{link}}) \setminus \bigcup_{i<j} V(P_i)$ such that $\phi(x_jv_1), \phi(v_1v) \notin \bigcup_{i<j} \phi(P_i)$, and since $j/n \leq \delta \ll \gamma$, we may let v_1 be such a vertex. Similarly, by (f), there is a vertex $v_2 \in (N_{G_2}(y_j) \cap N_{G_2}(v) \cap V'_{\text{link}}) \setminus \bigcup_{i<j} V(P_i)$ such that $\phi(y_jv_2), \phi(v_2v) \notin \bigcup_{i<j} \phi(P_i) \cup \{\phi(x_jv_1), \phi(v_1v)\}$. Now there is a reserve link P_j with ends x_j and y_j and internal vertices v, v_1, and v_2 that is vertex- and colour-disjoint from $\bigcup_{i<j} P_i$, as claimed, and therefore there exists a collection $\mathcal{P}_2$ of reserve links that links M'. Now $\mathcal{P}_1 \cup \mathcal{P}_2$ links M, so (5.5.1) holds. By (a), and since $\mathcal{M}$ is $(\delta, \mathcal{F})$-perfect, (5.5.2) holds, as required. □

5.5. Proof.
We now have all the tools we need to prove Lemma 3.9.

Proof of Lemma 3.9.   Consider an absorber partition of V, C, and K_n. By Lemmas 5.3, 5.4, and 5.5, there exists an outcome of the absorber partition satisfying the conclusions of these lemmas simultaneously. In particular, by Lemmas 5.3 and 5.4 there exists $H, \mathcal{A}$, and M such that, writing $(V_{\text{res}}, C_{\text{res}})$ for the bipartition of H,

- $H \cong \tilde{H}$ is a 36γ-absorbing template satisfying (5.3.1) and (5.3.2),
- $\mathcal{A}$ and H satisfy (5.4.1), and
- M satisfies (5.4.2).

Write $\mathcal{A} = \{A_1, \ldots, A_k\}$ and $E(M) = \{a_1b_1, \ldots, a_\ell b_\ell\}$. Consider $M_1 \dot\cup M_2 \dot\cup M_3 \dot\cup M_4$, where M_i is a matching satisfying (Mi) for $i \in [4]$ (see Section 5.1). By (5.4.2) we have $|V_{\text{abs}} \setminus V(M_1 \cup \cdots \cup M_4)| \leq 5\varepsilon n + 1 \leq 6\varepsilon n$. Thus by Lemma 5.5 there exist collections of links $\mathcal{P}$ and $\mathcal{T}$ in G' such that

- $\mathcal{P} \cup \mathcal{T}$ is a collection of links satisfying (5.5.1) with respect to $\bigcup_{i=1}^4 M_i$ and
- $\mathcal{P} \cup \mathcal{T}$ satisfies (5.5.2).

In particular $\mathcal{P}$ links $\bigcup_{i=1}^3 M_i$ and $\mathcal{T}$ links M_4. Let $T := M \cup \bigcup_{P \in \mathcal{T}} P$. By Fact 5.2, $(\mathcal{A}, \mathcal{P}, T, H)$ is a 36γ-absorber, as desired. Moreover, since H satisfies (5.3.1) and (5.3.2), M satisfies (5.4.2),

and $\mathcal{P} \cup \mathcal{T}$ satisfies (5.5.2), we have $\bigcup_{A \in \mathcal{A}} A \cup \bigcup_{P \in \mathcal{P}} P \cup T$ is contained in (V', C', G') with γ-bounded remainder, as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6. Finding many well-spread absorbing gadgets

The aim of this section is to prove Lemma 3.8, which states that, for appropriate μ, ε, almost all 1-factorizations of K_n are ε-locally edge-resilient and μ-robustly gadget-resilient. We will use switchings in $\mathcal{G}_D^{\mathrm{col}}$ for appropriate $D \subsetneq [n-1]$ to analyse the probability that a uniformly random $G \in \mathcal{G}_D^{\mathrm{col}}$ satisfies the necessary properties, and then use a 'weighting factor' (see Corollary 6.12) to make comparisons to the probability space corresponding to a uniform random choice of $G \in \mathcal{G}_{[n-1]}^{\mathrm{col}}$.

6.1. **Switchings.** We begin by analysing the property of ε-local edge-resilience.

Lemma 6.1. *Suppose $1/n \ll \varepsilon \ll 1$, and let $D \subseteq [n-1]$ have size $|D| = \varepsilon n$. Suppose $\mathbf{G} \in \mathcal{G}_D^{\mathrm{col}}$ is chosen uniformly at random. Then $\mathbb{P}\left[\mathbf{G} \text{ is } \varepsilon\text{-locally edge-resilient}\right] \geq 1 - \exp(-\varepsilon^3 n^2/1000)$.*

Proof. Note that if $G \in \mathcal{G}^{\mathrm{col}}$ has at least $\varepsilon^3 n^2/100$ edges with endpoints in V' for all choices of $V' \subseteq V$ of size precisely εn, then G is ε-locally edge-resilient. Fix $V' \subseteq V$ of size precisely εn. For any $G \in \mathcal{G}_D^{\mathrm{col}}$, we say that a subgraph $H \subseteq G$ together with a labelling of its vertices $V(H) = \{u, v, w, x, y, z\}$ is a *spin system* of G if $E(H) = \{vw, xy, zu\}$, where $u, v \in V'$, $w, x, y, z \in V \setminus V'$, $uv, wx, yz \notin E(G)$, and $\phi_G(vw) = \phi_G(xy) = \phi_G(zu)$. (Note that different labellings of a subgraph $H \subseteq G$ that both satisfy these conditions will be considered to correspond to different spin systems of G.) We now define the *spin* switching operation. Suppose $G \in \mathcal{G}_D^{\mathrm{col}}$ and $H \subseteq G$ is a spin system. Then we define $\mathrm{spin}_H(G)$ to be the coloured graph obtained from G by deleting the edges vw, xy, zu, and adding the edges uv, wx, yz, each with colour $\phi_G(vw)$. Writing $G' := \mathrm{spin}_H(G)$, we have $G' \in \mathcal{G}_D^{\mathrm{col}}$ and $e_{V',D}(G') = e_{V',D}(G) + 1$.

We define a partition $\{M_s\}_{s=0}^{\binom{\varepsilon n}{2}}$ of $\mathcal{G}_D^{\mathrm{col}}$ by setting $M_s := \{G \in \mathcal{G}_D^{\mathrm{col}} : e_{V',D}(G) = s\}$, for each $s \in [\binom{\varepsilon n}{2}]_0$. For each $s \in [\binom{\varepsilon n}{2} - 1]_0$ we define an auxiliary bipartite multigraph B_s with vertex bipartition (M_s, M_{s+1}), where for each $G \in M_s$ and each spin system $H \subseteq G$ we put an edge in B_s with endpoints $G \in M_s$ and $\mathrm{spin}_H(G) \in M_{s+1}$. Define $\delta_s := \min_{G \in M_s} d_{B_s}(G)$ and $\Delta_{s+1} := \max_{G \in M_{s+1}} d_{B_s}(G)$. Observe, by double counting $e(B_s)$, that $|M_s|/|M_{s+1}| \leq \Delta_{s+1}/\delta_s$. To bound Δ_{s+1} from above, we fix $G' \in M_{s+1}$ and bound the number of pairs (G, H), where $G \in M_s$ and H is a spin system of G such that $\mathrm{spin}_H(G) = G'$. There are $s + 1$ choices for the edge $e \in E_{V',D}(G')$ created by a spin operation, and 2 choices for which endpoint of e played the role of u in a spin, and which played the role of v. Now there are at most $(n/2)^2$ choices for two edges with colour $\phi_{G'}(e)$ in G' with both endpoints outside of V', and at most 8 choices for which endpoints of these edges played the roles of w, x, y, z in a spin operation yielding G'. We deduce that $\Delta_{s+1} \leq 4(s+1)n^2$.

Suppose that $s \leq \varepsilon^3 n^2/80$. To bound δ_s from below, we fix $G \in M_s$ and find a lower bound for the number of spin systems $H \subseteq G$. For a vertex $v \in V'$, let $D_G^*(v) \subseteq D$ denote the set of colours $c \in D$ such that the c-neighbour of v is not in V', in G. Let $V_G^* := \{v \in V' : |D_G^*(v)| \geq 9\varepsilon n/10\}$, and suppose for a contradiction that $|V_G^*| < 9\varepsilon n/10$. Then there are at least $\varepsilon n/10$ vertices $v \in V'$ for which there are at least $\varepsilon n/10$ colours $c \in D$ such that the c-neighbour of v is in V', in G, whence $s = e_{V',D}(G) \geq \varepsilon^2 n^2/200 > \varepsilon^3 n^2/80 \geq s$, a contradiction. Note further that, since $s \leq \varepsilon^3 n^2/80$, there are at least $\binom{9\varepsilon n/10}{2} - \varepsilon^3 n^2/80 \geq \varepsilon^2 n^2/4$ pairs $\{a, b\} \in \binom{V_G^*}{2}$ such that $ab \notin E(G)$. For each such choice of $\{a, b\}$, there are two choices of which vertex will play the role of u and which will play the role of v in a spin system. Since $u, v \in V_G^*$, there are at least $4\varepsilon n/5$ colours $c \in D$ such that the c-neighbour z of u, and the c-neighbour w of v, are such that $w, z \in V \setminus V'$, in G. Finally, there are at least $n/2 - 3\varepsilon n \geq n/4$ edges coloured c in G with neither endpoint in $V' \cup N_G(w) \cup N_G(z)$, and two choices of which endpoint of such an edge will play the role of x, and which will play the role of y. We deduce that $\delta_s \geq \varepsilon^3 n^4/5$. Altogether, we conclude that if $s \leq \varepsilon^3 n^2/80$ and M_s is non-empty, then M_{s+1} is non-empty and $|M_s|/|M_{s+1}| \leq 20(s+1)n^2/\varepsilon^3 n^4 \leq 1/2$.

Now, fix $s \leq \varepsilon^3 n^2/100$. If M_s is empty, then $\mathbb{P}\left[e_{V',D}(\mathbf{G}) = s\right] = 0$. If M_s is non-empty, then

$$\mathbb{P}\left[e_{V',D}(\mathbf{G}) = s\right] = \frac{|M_s|}{|\mathcal{G}_D^{\mathrm{col}}|} \leq \frac{|M_s|}{|M_{\varepsilon^3 n^2/80}|} = \prod_{j=s}^{\varepsilon^3 n^2/80 - 1} \frac{|M_j|}{|M_{j+1}|} \leq \left(\frac{1}{2}\right)^{\varepsilon^3 n^2/80 - s},$$

and thus

$$\mathbb{P}\left[e_{V',D}(\mathbf{G}) \leq \varepsilon^3 n^2/100\right] \leq \sum_{s=0}^{\varepsilon^3 n^2/100} \exp(-(\varepsilon^3 n^2/80 - s)\ln 2) \leq \exp\left(-\frac{\varepsilon^3 n^2}{800}\right).$$

A union bound over all choices of $V' \subseteq V$ of size εn now completes the proof. $\qquad\square$

We now turn to showing that for suitable $D \subseteq [n-1]$, almost all $G \in \mathcal{G}_D^{\mathrm{col}}$ are robustly gadget-resilient, which turns out to be a much harder property to analyse than local edge-resilience, and we devote the rest of this section to it. We first need to show that almost all $G \in \mathcal{G}_D^{\mathrm{col}}$ are 'quasirandom', in the sense that small sets of vertices do not have too many crossing edges.

Definition 6.2. Let $D \subseteq [n-1]$. We say that $G \in \mathcal{G}_D^{\mathrm{col}}$ is *quasirandom* if for all sets $A, B \subseteq V$, not necessarily distinct, such that $|A| = |B| = |D|$, we have that $e_G(A, B) < 8(|D| - 1)^3/n$. We define $\mathcal{Q}_D^{\mathrm{col}} := \{G \in \mathcal{G}_D^{\mathrm{col}} : G \text{ is quasirandom}\}$.

When we are analysing switchings to study the property of robust gadget-resilience (see Lemma 6.8), it will be important to condition on this quasirandomness. One can use another switching argument to show that almost all $G \in \mathcal{G}_D^{\mathrm{col}}$ are quasirandom.

Lemma 6.3. *Suppose that $1/n \ll \mu \ll 1$, let $D \subseteq [n-1]$ have size $|D| = \mu n + 1$. Suppose that $\mathbf{G} \in \mathcal{G}_D^{col}$ is chosen uniformly at random. Then $\mathbb{P}\left[\mathbf{G} \in \mathcal{Q}_D^{col}\right] \geq 1 - \exp(-\mu^3 n^2)$.*

Proof. Fix $A, B \subseteq V$ satisfying $|A| = |B| = \mu n + 1$. For any $G \in \mathcal{G}_D^{\mathrm{col}}$, we say that a subgraph $H \subseteq G$ together with a labelling of its vertices $V(H) = \{a, b, v, w\}$ is a *rotation system* of G if $E(H) = \{ab, vw\}$, where $a \in A$, $b \in B$, $v, w \notin A \cup B$, $aw, bv \notin E(G)$, and $\phi_G(ab) = \phi_G(vw)$. We now define the *rotate* switching operation. Suppose $G \in \mathcal{G}_D^{\mathrm{col}}$ and $H \subseteq G$ is a rotation system. Then we define $\mathrm{rot}_H(G)$ to be the coloured graph obtained from G by deleting the edges ab, vw, and adding the edges aw, bv, each with colour $\phi_G(ab)$. Writing $G' := \mathrm{rot}_H(G)$, notice that $G' \in \mathcal{G}_D^{\mathrm{col}}$ and $e_{G'}(A, B) = e_G(A, B) - 1$.

Lemma 6.3 follows by analysing the degrees of auxiliary bipartite multigraphs B_s in a similar way as in the proof of Lemma 6.1. We omit the details. $\qquad\square$

Next we will use a switching argument to find a large set of well-spread absorbing gadgets (cf. Definition 3.7). For this, we consider slightly more restrictive substructures than the absorbing gadgets defined in Definition 3.1. These additional restrictions (an extra edge f as well as an underlying partition $\mathcal{P}$ of the colours) give us better control over the switching process: they allow us to argue that we do not create more than one additional gadget per switch. Let $D \subseteq [n-1]$, $c \in [n-1] \setminus D$, write $D^* := D \cup \{c\}$, and let $G \in \mathcal{G}_{D^*}^{\mathrm{col}}$. Suppose that $\mathcal{P} = \{D_i\}_{i=1}^4$ is an (ordered) partition of D into four subsets, and let $x \in V$.

Definition 6.4. An $(x, c, \mathcal{P})$-*gadget* in G is a subgraph $J = A \cup \{f\}$ of G the following form (see Figure 4):
  (i) A is an (x, c)-absorbing gadget in G;
  (ii) there is an edge $e_1 \in \partial_A(x)$ such that $\phi(e_1) \in D_1$, and the remaining edge $e_2 \in \partial_A(x)$ satisfies $\phi(e_2) \in D_2$;
  (iii) the edge e_3 of A which is not incident to x but shares an endvertex with e_1 and an endvertex with e_2 satisfies $\phi(e_3) \in D_3$;
  (iv) $f = xv$ is an edge of G, where v is the unique vertex of A such that $\phi(\partial_A(v)) = \{c, \phi(e_1)\}$;
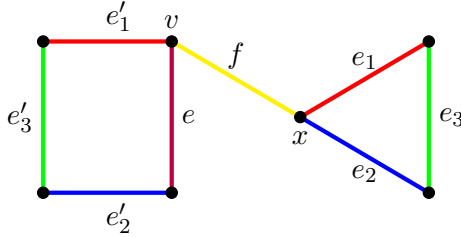 (v) $\phi(f) \in D_4$.

FIGURE 4. An $(x, c, \mathcal{P})$-gadget. Here, $\phi(f) \in D_4$, $\phi(e) = c$, and $\phi(e_i) = \phi(e_i') \in D_i$ for each $i \in [3]$.

We now define some terminology that will be useful for analysing how many $(x, c, \mathcal{P})$-gadgets there are in a graph $G \in \mathcal{G}_{D^*}^{\mathrm{col}}$, and how well-spread these gadgets are. Each of the terms we define here will have a dependence on the choice of the triple $(x, c, \mathcal{P})$, but since this triple will always be clear from context, for presentation we omit the $(x, c, \mathcal{P})$-notation.

Definition 6.5. We say that an $(x, c, \mathcal{P})$-gadget J in G is *distinguishable* in G if the edges e_3, e_3' of J such that $\phi(e_3) = \phi(e_3') \in D_3$ are such that there is no other $(x, c, \mathcal{P})$-gadget $J' \neq J$ in G such that $e_3 \in E(J')$ or $e_3' \in E(J')$.

We will aim only to count distinguishable $(x, c, \mathcal{P})$-gadgets, which will ensure the collection of gadgets we find is well-spread across the set of edges in $G \in \mathcal{G}_{D^*}^{\mathrm{col}}$ that can play the roles of e_3, e_3'. We also need to ensure that the collection of gadgets we find is well-spread across the c-edges of G.

Definition 6.6.
- For each c-edge e of $G \in \mathcal{G}_{D^*}^{\mathrm{col}}$, we define the *saturation* of e in G, denoted $\mathrm{sat}_G(e)$, or simply $\mathrm{sat}(e)$ when G is clear from context, to be the number of distinguishable $(x, c, \mathcal{P})$-gadgets of G which contain e. We say that e is *unsaturated* in G if $\mathrm{sat}(e) \leq |D| - 1$, *saturated* if $\mathrm{sat}(e) \geq |D|$, and *supersaturated* if $\mathrm{sat}(e) \geq |D| + 6$. We define $\mathrm{Sat}(G)$ to be the set of saturated c-edges of G, and $\mathrm{Unsat}(G) := E_c(G) \setminus \mathrm{Sat}(G)$.
- We define the function $r \colon \mathcal{G}_{D^*}^{\mathrm{col}} \to [n|D|/2]_0$ by

$$r(G) := |D||\mathrm{Sat}(G)| + \sum_{e \in \mathrm{Unsat}(G)} \mathrm{sat}(e).$$

In Lemma 6.8, we will use switchings to show that $r(G)$ is large (for some well-chosen $\mathcal{P}$) in almost all quasirandom $G \in \mathcal{G}_{D^*}^{\mathrm{col}}$. In Lemma 6.9, we use distinguishability, saturation, and the fact that any non-x vertex in an $(x, c, \mathcal{P})$-gadget must be incident to an edge playing the role of either e_3, e_3', or the c-edge, to show that $r(G)$ being large means that there are many well-distributed $(x, c, \mathcal{P})$-gadgets in G, and thus many well-spread (x, c)-absorbing gadgets. We now define a relaxation of $\mathcal{Q}_{D^*}^{\mathrm{col}}$, which will be a convenient formulation for ensuring that quasirandomness is maintained when we use switchings to find $(x, c, \mathcal{P})$-gadgets. For each $s \in [n|D|/2]_0$, we write $A_s^{D^*} := \{G \in \mathcal{G}_{D^*}^{\mathrm{col}} : r(G) = s\}$, and

$$Q_s^{D^*} := \{G \in \mathcal{G}_{D^*}^{\mathrm{col}} : e_G(A, B) < 8|D|^3/n + 6s \text{ for all } A, B \subseteq V \text{ such that } |A| = |B| = |D|\}.$$

We also define $T_s^{D^*} := A_s^{D^*} \cap Q_s^{D^*}$ and $\widetilde{\mathcal{Q}}_{D^*}^{\mathrm{col}} := \bigcup_{s=0}^{n|D|/2} T_s^{D^*}$. Notice that

$$(6.1) \qquad\qquad\qquad\qquad \mathcal{Q}_{D^*}^{\mathrm{col}} \subseteq \widetilde{\mathcal{Q}}_{D^*}^{\mathrm{col}}.$$

Finally, we discuss the switching operation that we will use in Lemma 6.8.

Definition 6.7. For any $G \in \mathcal{G}_{D^*}^{\mathrm{col}}$, we say that a subgraph $H \subseteq G$ together with a labelling of its vertices $V(H) = \{x, u_1, u_2, \ldots, u_{14}\}$ is a *twist system* of G if (see Figure 5):
(i) $E(H) = \{u_1u_2, u_3u_5, u_4u_6, u_5u_7, u_6u_8, u_7u_8, u_7x, xu_9, xu_{10}, u_9u_{11}, u_{10}u_{12}, u_{13}u_{14}\}$;
(ii) $\phi(u_5u_7) = \phi(xu_9) \in D_1$;
(iii) $\phi(u_6u_8) = \phi(xu_{10}) \in D_2$;
(iv) $\phi(u_1u_2) = \phi(u_3u_5) = \phi(u_4u_6) = \phi(u_9u_{11}) = \phi(u_{10}u_{12}) = \phi(u_{13}u_{14}) \in D_3$;
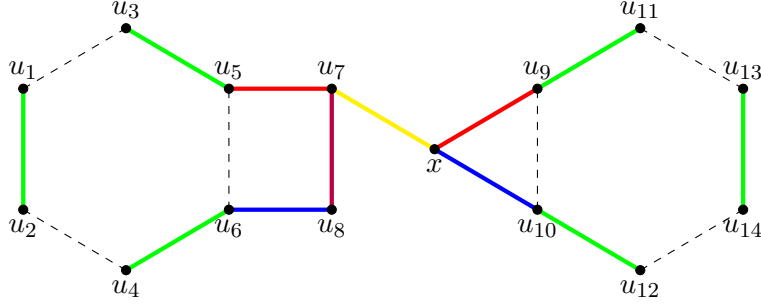(v) $\phi(u_7x) \in D_4$;

FIGURE 5. A twist system of G. Here, dashed edges represent non-edges of G, and the colours of the edges satisfy (ii)–(vi) in the definition of twist system.

(vi) $\phi(u_7u_8) = c$;

(vii) $u_1u_3, u_2u_4, u_5u_6, u_9u_{10}, u_{11}u_{13}, u_{12}u_{14} \notin E(G)$.

For a twist system H of G, we define $\text{twist}_H(G)$ to be the coloured graph obtained from G by deleting the edges u_1u_2, u_3u_5, u_4u_6, u_9u_{11}, $u_{10}u_{12}$, $u_{13}u_{14}$, and adding the edges u_1u_3, u_2u_4, u_5u_6, u_9u_{10}, $u_{11}u_{13}$, $u_{12}u_{14}$, each with colour $\phi_G(u_1u_2)$. The $(x, c, \mathcal{P})$-gadget in $\text{twist}_H(G)$ with edges u_5u_6, u_5u_7, u_6u_8, u_7u_8, u_7x, xu_9, xu_{10}, u_9u_{10} is called the *canonical* $(x, c, \mathcal{P})$-*gadget* of the twist.

We simultaneously switch two edges into the positions u_5u_6 and u_9u_{10} because it is much easier to find structures as in Figure 5 than it is to find such a structure with one of these edges already in place. Moreover, the two 'switching cycles' we use have three edges and three non-edges (rather than two of each, as in the rotation switching) essentially because of the extra freedom this gives us when choosing the edges u_1u_2 and $u_{13}u_{14}$. This extra freedom allows us to ensure that in almost all twist systems, one avoids undesirable issues like inadvertently creating more than one new gadget when one performs the twist.

The proof of Lemma 6.8 proceeds with a similar strategy to those of Lemmas 6.1 and 6.3, but it is much more challenging this time to show that graphs with low $r(G)$-value admit many ways to switch to yield a graph $G' \in \mathcal{G}_{D^*}^{\text{col}}$ satisfying $r(G') = r(G) + 1$.

Lemma 6.8. *Suppose that $1/n \ll \mu \ll 1$, and let $D \subseteq [n-1]$ have size $|D| = \mu n$. Let $x \in V$, let $c \in [n-1] \setminus D$, and let $\mathcal{P} = \{D_i\}_{i=1}^4$ be an equitable partition of D. Suppose that $\mathbf{G} \in \mathcal{G}_{D \cup \{c\}}^{\text{col}}$ is chosen uniformly at random. Then*

$$\mathbb{P}\left[r(\mathbf{G}) \leq \frac{\mu^4 n^2}{2^{23}} \;\middle|\; \mathbf{G} \in \widetilde{\mathcal{Q}}_{D \cup \{c\}}^{\text{col}}\right] \leq \exp\left(-\frac{\mu^4 n^2}{2^{24}}\right).$$

Proof. Write $D^* \coloneqq D \cup \{c\}$. Consider the partition $\{T_s^{D^*}\}_{s=0}^{nk/2}$ of $\widetilde{\mathcal{Q}}_{D^*}^{\text{col}}$, where $k \coloneqq |D|$. For each $s \in [nk/2 - 1]_0$, we define an auxiliary bipartite multigraph B_s with vertex bipartition $(T_s^{D^*}, T_{s+1}^{D^*})$ and an edge between G and $\text{twist}_H(G)$ whenever:
(a) $G \in T_s^{D^*}$;
(b) H is a twist system in G for which $G' \coloneqq \text{twist}_H(G) \in T_{s+1}^{D^*}$ and G' satisfies $\text{sat}_{G'}(e) = \text{sat}_G(e) + 1 \leq k$ for the c-edge $e = u_7u_8$ of H, with the canonical $(x, c, \mathcal{P})$-gadget of the twist G' being the only additional distinguishable $(x, c, \mathcal{P})$-gadget using this c-edge.

Define $\delta_s \coloneqq \min_{G \in T_s^{D^*}} d_{B_s}(G)$ and $\Delta_{s+1} \coloneqq \max_{G \in T_{s+1}^{D^*}} d_{B_s}(G)$. Thus $|T_s^{D^*}|/|T_{s+1}^{D^*}| \leq \Delta_{s+1}/\delta_s$. To bound Δ_{s+1} from above, we fix $G' \in T_{s+1}^{D^*}$ and bound the number of pairs (G, H), where $G \in T_s^{D^*}$ and H is a twist system of G such that $\text{twist}_H(G) = G'$ and (b) holds. Firstly, note that

$$\sum_{\substack{e \in E_c(G') \\ \text{sat}_{G'}(e) \leq k}} \text{sat}_{G'}(e) \leq r(G') = s + 1.$$

Thus, it follows from condition (b) that there are at most $s+1$ choices for the canonical $(x, c, \mathcal{P})$-gadget of a twist yielding G' for which we record an edge in B_s. Fixing this $(x, c, \mathcal{P})$-gadget

fixes the vertices of V which played the roles of x, u_5, u_6, ..., u_{10} in a twist yielding G'. To determine all possible sets of vertices playing the roles of u_1, u_2, u_3, u_4, u_{11}, u_{12}, u_{13}, u_{14} (thus determining H and G such that $\mathrm{twist}_H(G) = G'$), it suffices to find all choices of four edges of G' with colour $\phi_{G'}(u_5 u_6)$ satisfying the necessary non-adjacency conditions. There are at most $(n/2)^4$ choices for these four edges, and at most $4! \cdot 2^4$ choices for which endpoints of these edges play which role. We deduce that $\Delta_{s+1} \leq 24 n^4 (s+1)$.

Suppose that $s \leq k^4/2^{22} n^2$. To bound δ_s from below, we fix $G \in T_s^{D^*}$ and find a lower bound for the number of twist systems $H \subseteq G$ for which we record an edge between G and $\mathrm{twist}_H(G)$ in B_s. To do this, we will show that there are many choices for a set of four colours and two edges, such that each of these sets uniquely identifies a twist system in G for which we record an edge in B_s. Note that since $s \leq k^4/2^{22} n^2$ and $G \in Q_s^{D^*}$, we have

$$(6.2) \qquad e_G(A, B) \leq 10 k^3/n \quad \text{for all sets } A, B \subseteq V \text{ of sizes } |A| = |B| = k.$$

We begin by finding subsets of D_3 and D_4 with some useful properties in G.

Claim 1: There is a set $D_3^{good} \subseteq D_3$ of size $|D_3^{good}| \geq k/8$ such that for all $d \in D_3^{good}$ we have
  (i) $|E_d(N_{D_1}(x), N_{D_2}(x))| \leq 200 k^2/n$;
  (ii) *there are at most $64 k^3/n^2$ d-edges e in G with the property that e lies in some distinguishable $(x, c, \mathcal{P})$-gadget in G whose c-edge is not supersaturated.*

Proof of claim: Observe that $|N_{D_1}(x)| = |N_{D_2}(x)| = k/4$. Then, by (arbitrarily extending $N_{D_1}(x)$, $N_{D_2}(x)$ and) applying (6.2), we see that $e(N_{D_1}(x), N_{D_2}(x)) \leq 10 k^3/n$. Thus there is a set $\hat{D}_3 \subseteq D_3$ of size $|\hat{D}_3| \geq 3k/16$ such that each $d \in \hat{D}_3$ satisfies (i). Next, notice that, since $r(G) = s$, there are at most $s/k \leq k^3/2^{22} n^2$ saturated c-edges in G. Suppose for a contradiction that at least $k/16$ colours $d \in D_3$ are such that there are at least $64 k^3/n^2$ d-edges e in G with the property that e lies in some distinguishable $(x, c, \mathcal{P})$-gadget in G whose c-edge is not supersaturated. Then, by considering the contribution of these distinguishable $(x, c, \mathcal{P})$-gadgets to $r(G)$, and accounting for saturated c-edges, we obtain that $r(G) \geq (k/16) \cdot 32 k^3/n^2 - 5 k^3/2^{22} n^2 > s$, a contradiction. Thus there is a set $\tilde{D}_3 \subseteq D_3$ of size $|\tilde{D}_3| \geq 3k/16$ such that each $d \in \tilde{D}_3$ satisfies (ii). We define $D_3^{good} := \hat{D}_3 \cap \tilde{D}_3$, and note that $|D_3^{good}| \geq k/8$.
—

We also define $D_4^{good} \subseteq D_4$ to be the set of colours $d_4 \in D_4$ such that the c-edge e incident to the d_4-neighbour of x in G satisfies $\mathrm{sat}(e) \leq k-1$. Observe that $|D_4^{good}| \geq k/8$, since otherwise there are at least $k/16$ saturated c-edges in G, whence $r(G) \geq k^2/16 > s$, a contradiction.

We now show that there are many choices of a vector $(d_1, d_2, d_3, d_4, \overrightarrow{f_1}, \overrightarrow{f_2})$ where each $d_i \in D_i$ and each $\overrightarrow{f_j}$ is an edge $f_j \in E_{d_3}(G)$ together with an identification of which endpoints will play which role, such that each vector uniquely gives rise to a candidate of a twist system $H \subseteq G$. We can begin to construct such a candidate by choosing $d_4 \in D_4^{good}$ and letting u_7 denote the d_4-neighbour of x in G, and letting u_8 denote the c-neighbour of u_7. Secondly, we choose $d_1 \in D_1$, avoiding the colour of the edge $x u_8$ (if it is present), and let u_5 denote the d_1-neighbour of u_7, and let u_9 denote the d_1-neighbour of x. Next, we choose $d_2 \in D_2$, avoiding the colours of the edges $u_5 u_8$, $u_5 x$, $u_8 x$, $u_8 u_9$ in G (if they are present), and let u_6 denote the d_2-neighbour of u_8, and let u_{10} denote the d_2-neighbour of x. Then, we choose $d_3 \in D_3^{good}$, avoiding the colours of all edges in $E_G(\{x, u_5, u_6, \ldots, u_{10}\})$. We let u_3, u_4, u_{11}, u_{12} denote the d_3-neighbours of u_5, u_6, u_9, u_{10}, respectively. Finally, we choose two distinct edges $f_1, f_2 \in E_{d_3}(G)$ which are not incident to any vertex in $\{x, u_3, u_4, \ldots, u_{12}\}$, and we choose which endpoint of f_1 will play the role of u_1 and which will play the role of u_2, and choose which endpoint of f_2 will play the role of u_{13} and which will play the role of u_{14}. Let Λ denote the set of all possible vectors $(d_1, d_2, d_3, d_4, \overrightarrow{f_1}, \overrightarrow{f_2})$ that can be chosen in this way, so that $|\Lambda| \geq \frac{k}{8} \cdot \frac{3k}{16} \cdot \frac{k}{8} \cdot \frac{k}{16} \cdot \frac{n}{4} \cdot 2 \cdot \frac{n}{4} \cdot 2 = 3 k^4 n^2 / 2^{16}$. Further, let $H(\lambda) \subseteq G$ denote the labelled subgraph of G corresponding to $\lambda \in \Lambda$ in the above way. If $H(\lambda)$ is a twist system, then we sometimes say that we 'twist on λ' to mean that we perform the twist operation to obtain $\mathrm{twist}_{H(\lambda)}(G)$ from G.

It is clear that $H(\lambda)$ is unique for all vectors $\lambda \in \Lambda$, and that $H(\lambda)$ satisfies conditions (i)–(vi) of the definition of a twist system. However, some $H(\lambda)$ may fail to satisfy (vii), and some may fail to satisfy condition (b) in the definition of adjacency in B_s. We now show that only for a small proportion of $\lambda \in \Lambda$ do either of these problems occur. We begin by ensuring that most $\lambda \in \Lambda$ give rise to twist systems.

Claim 2: There is a subset $\Lambda_1 \subseteq \Lambda$ such that $|\Lambda_1| \geq 9|\Lambda|/10$ and $H(\lambda)$ is a twist system for all $\lambda \in \Lambda_1$.

Proof of claim: Fix any choice of $d_3 \in D_3^{\mathrm{good}}$, $d_4 \in D_4^{\mathrm{good}}$ and $\overrightarrow{f_1}, \overrightarrow{f_2}$ appearing concurrently in some $\lambda \in \Lambda$, and note that there are at most $(k/4)^2 \cdot n^2$ such choices. Here and throughout the remainder of the proof of Lemma 6.8, we write u_7 for the d_4-neighbour of x, we write u_8 for the c-neighbour of u_7, and so on, where the choice of $d_1, d_2, d_3, d_4, \overrightarrow{f_1}, \overrightarrow{f_2}$ will always be clear from context. Note that fixing d_3, d_4 only fixes the vertices x, u_7, u_8. There are at most $10k^3/n$ pairs (d_1, d_2) with each $d_i \in D_i$ such that there is an edge $u_5u_6 \in E(G)$, since otherwise $e(N_{D_1}(u_7), N_{D_2}(u_8)) > 10k^3/n$, contradicting (6.2). Similarly, there are at most $10k^3/n$ pairs (d_1, d_2) with each $d_i \in D_i$ such that u_9u_{10} is an edge of G. We deduce that there are at most $(20k^3/n) \cdot (k/4)^2 \cdot n^2 = 5k^5n/4$ vectors $\lambda \in \Lambda$ for which $H(\lambda)$ is such that either u_5u_6 or u_9u_{10} is an edge of G. Now fix instead $d_1, d_2, d_3, d_4, \overrightarrow{f_2}$. Note that $|N_G(u_3) \cup N_G(u_4)| \leq 2k + 2$ so that there are at most $4k + 4$ choices of $\overrightarrow{f_1}$ such that either u_1u_3 or u_2u_4 is an edge of G. Analysing the pairs $u_{11}u_{13}$ and $u_{12}u_{14}$ similarly, we deduce that altogether, there are at most $5k^5n/4 + 2((k/4)^4 \cdot n \cdot (4k + 4)) \leq 2k^5n \leq |\Lambda|/10$ vectors $\lambda \in \Lambda$ for which $H(\lambda)$ fails to be a twist system. $-$

We now show that only for a small proportion of $\lambda \in \Lambda_1$ does $H(\lambda)$ fail to give rise to an edge in B_s, by showing that most $H(\lambda)$ satisfy the following properties:
(P1) $\mathrm{twist}_{H(\lambda)}(G) \in Q_{s+1}^{D^*}$;
(P2) Deletion of the six d_3-edges in $H(\lambda)$ does not decrease $r(G)$;
(P3) The canonical $(x, c, \mathcal{P})$-gadget of the twist $\mathrm{twist}_{H(\lambda)}(G)$ is distinguishable, and it is the only $(x, c, \mathcal{P})$-gadget which is in $\mathrm{twist}_{H(\lambda)}(G)$ but not in G.

Firstly, since $G \in Q_s^{D^*}$ and we only create six new edges in any twist, it is clear that $H(\lambda)$ satisfies (P1) for all $\lambda \in \Lambda_1$.

Claim 3: There is a subset $\Lambda_2 \subseteq \Lambda_1$ such that $|\Lambda_2| \geq 9|\Lambda_1|/10$ and $H(\lambda)$ satisfies property (P2) for all $\lambda \in \Lambda_2$.

Proof of claim: Fix $d_1 \in D_1$, $d_3 \in D_3^{\mathrm{good}}$, $d_4 \in D_4^{\mathrm{good}}$, $\overrightarrow{f_1}, \overrightarrow{f_2}$ appearing concurrently in some $\lambda \in \Lambda_1$. Let $F_{d_3}(G) \subseteq E_{d_3}(G)$ be the set of d_3-edges e in G with the property that e is in some distinguishable $(x, c, \mathcal{P})$-gadget in G whose c-edge is not supersaturated. Recall that $|F_{d_3}(G)| \leq 64k^3/n^2$ since $d_3 \in D_3^{\mathrm{good}}$. Observe then that there are at most $128k^3/n^2$ colours $d_2 \in D_2$ such that u_{10} is the endpoint of an edge in $F_{d_3}(G)$. Thus for all but at most $(k/4)^3 \cdot n^2 \cdot 128k^3/n^2 = 2k^6$ choices of $\lambda = (d_1, d_2, d_3, d_4, \overrightarrow{f_1}, \overrightarrow{f_2}) \in \Lambda_1$, the edge $u_{10}u_{12}$ is not in $F_{d_3}(G)$. Now fix instead $d_1, d_2, d_3, d_4, \overrightarrow{f_2}$. Then since $d_3 \in D_3^{\mathrm{good}}$, there are at most $128k^3/n^2$ choices of $\overrightarrow{f_1}$ such that $f_1 \in F_{d_3}(G)$, so that for all but at most $(k/4)^4 \cdot n \cdot 128k^3/n^2 = k^7/2n$ vectors $\lambda \in \Lambda_1$, $H(\lambda)$ is such that $f_1 \notin F_{d_3}(G)$. Similar analyses show that there are at most $8k^6 + k^7/n \leq 9k^6 \leq |\Lambda_1|/10$ choices of $\lambda \in \Lambda_1$ such that $\{u_1u_2, u_3u_5, u_4u_6, u_9u_{11}, u_{10}u_{12}, u_{13}u_{14}\} \cap F_{d_3}(G) \neq \emptyset$. By definition of $F_{d_3}(G)$ and supersaturation of a c-edge, we deduce that for all remaining $\lambda \in \Lambda_1$, $H(\lambda)$ is such that deleting the edges $u_1u_2, u_3u_5, u_4u_6, u_9u_{11}, u_{10}u_{12}, u_{13}u_{14}$ does not decrease $r(G)$. $-$

When we perform a twist operation on a twist system H in G, since the only new edges we add have some colour in D_3, we have that for any new distinguishable $(x, c, \mathcal{P})$-gadget J we create in the twist, one of the new edges $u_1u_3, u_2u_4, u_5u_6, u_9u_{10}, u_{11}u_{13}, u_{12}u_{14}$ of the twist is playing the role of either v_5v_6 or v_9v_{10} in J. (Here and throughout the rest of the proof, we imagine completed $(x, c, \mathcal{P})$-gadgets J as having vertices labelled $x, v_5, \ldots, v_{10}$, where the role of v_i corresponds to the role of u_i in Figure 5.) We now show that for most $\lambda \in \Lambda_2$, $H(\lambda)$

satisfies property (P3). This is the most delicate part of the argument, and we break it into three more claims.

Claim 4: There is a subset $\Lambda_3 \subseteq \Lambda_2$ such that $|\Lambda_3| \geq 9|\Lambda_2|/10$ and all $\lambda \in \Lambda_3$ are such that if J is an $(x, c, \mathcal{P})$-gadget that is in $\text{twist}_{H(\lambda)}(G)$ but not in G, then the pair $u_9 u_{10}$ of $H(\lambda)$ plays the role of $v_9 v_{10}$.

Proof of claim: Since the only edges added by any twist operation all have colour in D_3, it suffices to show that at most $|\Lambda_2|/10$ vectors $\lambda \in \Lambda_2$ are such that twisting on λ creates an $(x, c, \mathcal{P})$-gadget J for which either

    (i) one of the pairs $u_1 u_3$, $u_2 u_4$, $u_5 u_6$, $u_{11} u_{13}$, $u_{12} u_{14}$ of $H(\lambda)$ plays the role of $v_9 v_{10}$, or

    (ii) the edge $v_9 v_{10}$ of J is present in G.

To address (i), we show that $u_1, u_2, u_5, u_{11}, u_{12} \notin N_G(x)$ for all but at most $|\Lambda_2|/20$ vectors $\lambda \in \Lambda_2$. Note firstly that at most $10k^3/n$ pairs (d_1, d_4) where $d_1 \in D_1$, $d_4 \in D_4^{\text{good}}$ are such that $u_5 \in N_G(x)$, since otherwise $e(N_{D_4}(x), N_G(x)) > 10k^3/n$, contradicting (6.2). Thus, at most $(k/4)^2 \cdot n^2 \cdot 10k^3/n = 5k^5 n/8$ choices of $\lambda \in \Lambda_2$ are such that $u_5 \in N_G(x)$. Now fix d_1, d_2, d_3, d_4, $\overrightarrow{f_2}$ appearing concurrently in some $\lambda \in \Lambda_2$. Notice that there are at most $2k + 2$ choices of $\overrightarrow{f_1}$ such that f_1 has at least one endpoint in $N_G(x)$. Analysing $\overrightarrow{f_2}$ similarly, we deduce that there are at most $5k^5 n/8 + 2(k/4)^4(2k+2)n \leq |\Lambda_2|/20$ choices of $\lambda \in \Lambda_2$ such that at least one of u_1, u_2, u_5, u_{13}, u_{14} lies in $N_G(x)$.

Turning now to (ii), we show that at most $|\Lambda_2|/20$ vectors $\lambda \in \Lambda_2$ are such that twisting on λ creates any $(x, c, \mathcal{P})$-gadgets J for which the edge $v_9 v_{10}$ of J is present in G (and thus one of the pairs $u_1 u_3, u_2 u_4, u_5 u_6, u_9 u_{10}, u_{11} u_{13}, u_{12} u_{14}$ of $H(\lambda)$ plays the role of $v_5 v_6$). To do this, we use some of the properties of D_3^{good}. Fix $d_2 \in D_2$, $d_3 \in D_3^{\text{good}}$, $d_4 \in D_4^{\text{good}}$, $\overrightarrow{f_1}$, $\overrightarrow{f_2}$ appearing concurrently in some $\lambda \in \Lambda_3$. Note that since $d_3 \in D_3^{\text{good}}$, there are at most $200k^2/n$ pairs (d_1', d_2') where $d_1' \in D_1$, $d_2' \in D_2$, such that there is a K_3 in G with vertices x, w_1, w_2, where w_i is the d_i'-neighbour of x for $i \in \{1, 2\}$, and the edge $w_1 w_2$ is coloured d_3. Let the set of these pairs (d_1', d_2') be denoted $L(d_3)$. For each pair $\ell = (d_1', d_2') \in L(d_3)$, let z_ℓ^1 be the end of the $d_1' c d_2'$-walk starting at u_{10}. Similarly, let z_ℓ^2 denote the end of the $d_2' c d_1'$-walk starting at u_{10}. Define $M \coloneqq \bigcup_{\ell \in L(d_3)} \{z_\ell^1, z_\ell^2\}$, so that $|M| \leq 400k^2/n$. Since there are at most $400k^2/n$ choices of $d_1 \in D_1$ for which we obtain $u_9 \in M$, we deduce that for all but at most $(k/4)^3 \cdot n^2 \cdot 400k^2/n = 25k^5 n/4$ vectors $\lambda \in \Lambda_2$, $H(\lambda)$ is such that adding the edge $u_9 u_{10}$ in colour d_3 does not create a new $(x, c, \mathcal{P})$-gadget J where $u_9 u_{10}$ plays the role of $v_5 v_6$ in J and the edge playing the role of $v_9 v_{10}$ in J is already present in G before the twist. One can observe similarly that for all but at most $25k^5 n/4$ vectors $\lambda \in \Lambda$, $H(\lambda)$ is such that adding the edge $u_5 u_6$ in colour d_3 does not create a new $(x, c, \mathcal{P})$-gadget J where $u_5 u_6$ plays the role of $v_5 v_6$ in J and the edge playing the role of $v_9 v_{10}$ in J is already present in G before the twist.

Now fix instead d_1, d_2, d_3, d_4, $\overrightarrow{f_2}$ appearing concurrently in some $\lambda \in \Lambda_2$. For each $\ell = (d_1', d_2') \in L(d_3)$, let y_ℓ^1 be the end of the $d_1' c d_2'$-walk starting at u_4, let y_ℓ^2 be the end of the $d_2' c d_1'$-walk starting at u_4, let z_ℓ^1 be the end of the $d_1' c d_2'$-walk starting at u_3, and let z_ℓ^2 be the end of the $d_2' c d_1'$-walk starting at u_3. Define $M \coloneqq \bigcup_{\ell \in L(d_3)} \{y_\ell^1, y_\ell^2, z_\ell^1, z_\ell^2\}$, and notice that $|M| \leq 800k^2/n$. We deduce that there are at most $1600k^2/n$ choices of $\overrightarrow{f_1}$ such that f_1 has an endpoint in M, and that for all remaining choices of $\overrightarrow{f_1}$, twisting on $\lambda = (d_1, d_2, d_3, d_4, \overrightarrow{f_1}, \overrightarrow{f_2})$ cannot create a new $(x, c, \mathcal{P})$-gadget J where the new d_3-edges $u_1 u_3$ or $u_2 u_4$ play the role of $v_5 v_6$ in J and the edge $v_9 v_{10}$ of J is present in G. Analysing $\overrightarrow{f_2}$ similarly, we conclude that for all but at most $13k^5 n \leq |\Lambda_2|/20$ choices of $\lambda \in \Lambda_2$, twisting on λ cannot create a new $(x, c, \mathcal{P})$-gadget J for which the edge $v_9 v_{10}$ of J is present in G. $-$

Claim 5: There is a subset $\Lambda_4 \subseteq \Lambda_3$ such that $|\Lambda_4| \geq 9|\Lambda_3|/10$ and all $\lambda \in \Lambda_4$ are such that if J is an $(x, c, \mathcal{P})$-gadget that is in $\text{twist}_{H(\lambda)}(G)$ but not in G, then the pair $u_5 u_6$ of $H(\lambda)$ plays the role of $v_5 v_6$.

Proof of claim: By Claim 4, it will suffice to show that at most $|\Lambda_3|/10$ vectors $\lambda \in \Lambda_3$ are such that twisting on λ creates an $(x, c, \mathcal{P})$-gadget J for which either

(i) one of the pairs $u_1u_3, u_2u_4, u_{11}u_{13}, u_{12}u_{14}$ of $H(\lambda)$ plays the role of v_5v_6, and u_9u_{10} plays the role of v_9v_{10}, or

(ii) the edge v_5v_6 of J is present in G and the pair u_9u_{10} of $H(\lambda)$ plays the role of v_9v_{10}.

To address (i), fix $d_1, d_2, d_3, d_4, \overrightarrow{f_2}$ appearing concurrently in some $\lambda \in \Lambda_3$. Let a_1 be the end of the d_1cd_2-walk starting at u_4, let a_2 be the end of the d_2cd_1-walk starting at u_4, let b_1 be the end of the d_1cd_2-walk starting at u_3, and let b_2 be the end of the d_2cd_1-walk starting at u_3. Since there are at most 8 choices of $\overrightarrow{f_1}$ with an endpoint in $\{a_1, a_2, b_1, b_2\}$, we deduce that for all remaining choices of $\overrightarrow{f_1}$, twisting on $\lambda = (d_1, d_2, d_3, d_4, \overrightarrow{f_1}, \overrightarrow{f_2})$ cannot create an $(x, c, \mathcal{P})$-gadget J for which the new d_3-edges u_1u_3 or u_2u_4 play the role of v_5v_6 in J and u_9u_{10} plays the role of v_9v_{10}. Analysing $\overrightarrow{f_2}$ similarly, we conclude that we must discard at most $k^4n/16 \leq |\Lambda_3|/20$ vectors $\lambda \in \Lambda_3$ to account for (i).

Turning now to (ii), write $D_4 = \{d_4^1, d_4^2, \ldots, d_4^{k/4}\}$. For each $d_4^i \in D_4$, let y_i be the d_4^i-neighbour of x, let z_i be the c-neighbour of y_i, define $R_i := N_{D_1}(y_i)$ and $S_i := N_{D_2}(z_i)$. Notice that $\sum_{i=1}^{k/4} e(R_i, S_i) \leq 5k^4/2n$, since otherwise we obtain a contradiction to (6.2) for some pair (R_i, S_i). We deduce that there are at most $5k^4/2n$ triples (d_1, d_2, d_3) with each $d_i \in D_i$ for which adding the edge u_9u_{10} in colour d_3 creates an $(x, c, \mathcal{P})$-gadget J for which u_9u_{10} plays the role of v_9v_{10} in J and the edge playing the role of v_5v_6 is already present in G, whence at most $(5k^4/2n) \cdot (k/4) \cdot n^2 = 5k^5n/8 \leq |\Lambda_3|/20$ choices of $\lambda \in \Lambda_3$ are such that twisting on λ creates an $(x, c, \mathcal{P})$-gadget of this type. $-$

Claim 6: There is a subset $\Lambda_5 \subseteq \Lambda_4$ such that $|\Lambda_5| \geq 9|\Lambda_4|/10$ and all $\lambda \in \Lambda_5$ are such that if J is an $(x, c, \mathcal{P})$-gadget that is in $\mathrm{twist}_{H(\lambda)}(G)$ but not in G and the pairs u_5u_6, u_9u_{10} of $H(\lambda)$ play the roles of the edges v_5v_6, v_9v_{10} of J respectively, then J is the canonical $(x, c, \mathcal{P})$-gadget of the twist.

Proof of claim: Fix $d_3, d_4, \overrightarrow{f_1}, \overrightarrow{f_2}$ appearing concurrently in some $\lambda \in \Lambda_4$. By (6.2), we have that $e(N_{D_2}(u_8), N_{D_4}(x)) \leq 10k^3/n$. We deduce that there are at most $10k^3/n$ choices of the pair (d_1, d_2) such that the d_1-neighbour of u_6 lies in $N_{D_4}(x)$, whence for all but at most $5k^5n/8 \leq |\Lambda_4|/10$ choices of $\lambda \in \Lambda_4$, the canonical $(x, c, \mathcal{P})$-gadget of the twist is the only new $(x, c, \mathcal{P})$-gadget for which u_5u_6, u_9u_{10} play the roles of v_5v_6, v_9v_{10} respectively. $-$

Note that, by Claims 4–6, the canonical $(x, c, \mathcal{P})$-gadget of a twist on $\lambda \in \Lambda_5$ is clearly distinguishable in $\mathrm{twist}_{H(\lambda)}(G)$ since its edges v_5v_6 and v_9v_{10} with colours in D_3 were added by the twist and performing this twist creates no other $(x, c, \mathcal{P})$-gadgets. Thus Claims 4–6 imply that $H(\lambda)$ satisfies (P3) for all $\lambda \in \Lambda_5$. Recalling that $\mathrm{sat}_G(e) \leq k - 1$ for the c-edge e of $H(\lambda)$ for all $\lambda \in \Lambda$ and also using Claim 3, we now deduce that $r(\mathrm{twist}_{H(\lambda)}(G)) = r(G) + 1$, and thus $\mathrm{twist}_{H(\lambda)}(G) \in A_{s+1}^{D^*}$, for all $\lambda \in \Lambda_5$. Since $H(\lambda)$ satisfies (P1) for all $\lambda \in \Lambda_1$, we deduce that $\mathrm{twist}_{H(\lambda)}(G) \in T_{s+1}^{D^*}$ for all $\lambda \in \Lambda_5$, and that $\delta_s \geq |\Lambda_5| \geq |\Lambda|/2 \geq 3k^4n^2/2^{17}$. We conclude that if $s \leq k^4/2^{22}n^2$ and $T_s^{D^*}$ is non-empty, then $T_{s+1}^{D^*}$ is non-empty and $|T_s^{D^*}|/|T_{s+1}^{D^*}| \leq 2^{17} \cdot 24n^4(s+1)/3k^4n^2 \leq 1/2$. Now, fix $s \leq \mu^4n^2/2^{23}$. If $T_s^{D^*}$ is empty, then $\mathbb{P}\left[r(\mathbf{G}) = s \mid \mathbf{G} \in \widetilde{\mathcal{Q}}_{D^*}^{\mathrm{col}}\right] = 0$. If $T_s^{D^*}$ is non-empty, then

$$\mathbb{P}\left[r(\mathbf{G}) = s \mid \mathbf{G} \in \widetilde{\mathcal{Q}}_{D^*}^{\mathrm{col}}\right] = \frac{|T_s^{D^*}|}{|\widetilde{\mathcal{Q}}_{D^*}^{\mathrm{col}}|} \leq \frac{|T_s^{D^*}|}{|T_{k^4/2^{22}n^2}^{D^*}|} = \prod_{j=s}^{k^4/2^{22}n^2-1} \frac{|T_j^{D^*}|}{|T_{j+1}^{D^*}|} \leq (1/2)^{k^4/2^{22}n^2-s},$$

and thus,

$$\mathbb{P}\left[r(\mathbf{G}) \leq \mu^4n^2/2^{23} \mid \mathbf{G} \in \widetilde{\mathcal{Q}}_{D^*}^{\mathrm{col}}\right] \leq \sum_{s=0}^{\mu^4n^2/2^{23}} \exp(-(k^4/2^{22}n^2 - s)\ln 2) \leq \exp\left(-\frac{\mu^4n^2}{2^{24}}\right),$$

which completes the proof of the lemma. $\square$

Next, we show that in order to find many well-spread (x, c)-absorbing gadgets in $G \in \mathcal{G}^{\mathrm{col}}_{D \cup \{c\}}$, it suffices to show that $r(G)$ is large for some equitable partition $\mathcal{P}$ of D into four parts. (Recall that 'well-spread' was defined in Definition 3.7.)

Lemma 6.9. *Suppose that $1/n \ll \mu$, and let $D \subseteq [n-1]$ be such that $|D| \leq \mu n$. Let $x \in V$, let $c \in [n-1] \setminus D$, and let $\mathcal{P} = \{D_i\}_{i=1}^4$ be an equitable partition of D. Then for any integer $t \geq 0$ and any $G \in \mathcal{G}^{col}_{D \cup \{c\}}$, if $r(G) \geq t$, then G contains a $5\mu n/4$-well-spread collection of t distinct (x, c)-absorbing gadgets.*

Proof. Let $G \in \mathcal{G}^{\mathrm{col}}_{D \cup \{c\}}$, let $t \geq 0$ be an integer, and suppose that $r(G) \geq t$. Then, since $|D| \leq \mu n$ and by definition of r, we deduce that there is a collection $\mathcal{A}_{(x,c,\mathcal{P})}$ of t distinct $(x, c, \mathcal{P})$-gadgets satisfying the following conditions:

    (i) Each edge of G with colour in D_3 is contained in at most one $(x, c, \mathcal{P})$-gadget $J \in \mathcal{A}_{(x,c,\mathcal{P})}$;

    (ii) Each c-edge of G is contained in at most μn $(x, c, \mathcal{P})$-gadgets $J \in \mathcal{A}_{(x,c,\mathcal{P})}$.

Fix $v \in V \setminus \{x\}$. Let e be the c-edge of G incident to v and for each $d \in D_3$ let f_d be the d-edge of G incident to v. Then by conditions (i) and (ii) there are at most $5\mu n/4$ $(x, c, \mathcal{P})$-gadgets $J \in \mathcal{A}_{(x,c,\mathcal{P})}$ containing any of the edges in $\{e\} \cup \bigcup_{d \in D_3} \{f_d\}$. Note that if v is contained in some $J \in \mathcal{A}_{(x,c,\mathcal{P})}$, then v is incident to either the c-edge in J, or to one of the edges in J with colour in D_3. We thus conclude that v is contained in at most $5\mu n/4$ $(x, c, \mathcal{P})$-gadgets $J \in \mathcal{A}_{(x,c,\mathcal{P})}$. It immediately follows that no edge of G is contained in more than $5\mu n/4$ $(x, c, \mathcal{P})$-gadgets $J \in \mathcal{A}_{(x,c,\mathcal{P})}$.

For each $d \in D_1 \cup D_2 \cup D_4$, there are at most $5\mu n/4$ $J \in \mathcal{A}_{(x,c,\mathcal{P})}$ with $d \in \phi(J)$ since each such J must contain the d-neighbour of x in G. For each $d \in D_3$, there are at most $\mu n/2$ d-edges f in G such that both endpoints of f are neighbours of x. Any $J \in \mathcal{A}_{(x,c,\mathcal{P})}$ for which $d \in \phi(J)$ must contain one of these edges f. Thus by (i), there are at most $\mu n/2$ $J \in \mathcal{A}_{(x,c,\mathcal{P})}$ such that $d \in \phi(J)$.

Finally, define a function g on $\mathcal{A}_{(x,c,\mathcal{P})}$ by setting $g(J) := J - f$, where f is the unique edge of J with colour in D_4, for each $J \in \mathcal{A}_{(x,c,\mathcal{P})}$. Then it is clear that g is injective and that $g(J)$ is an (x, c)-absorbing gadget, for each $J \in \mathcal{A}_{(x,c,\mathcal{P})}$. Thus, $g(\mathcal{A}_{(x,c,\mathcal{P})})$ is a $5\mu n/4$-well-spread collection of t distinct (x, c)-absorbing gadgets in G, as required. $\qquad \square$

6.2. **Weighting factor.** We now state two results on the number of 1-factorizations in dense d-regular graphs G, where a 1-factorization of G consists of an ordered set of d perfect matchings in G. We will use these results to find a 'weighting factor' (see Corollary 6.12), which we will use to compare the probabilities of particular events occurring in different probability spaces. For any graph G, let $M(G)$ denote the number of distinct 1-factorizations of G, and for any $n, d \in \mathbb{N}$, let $\mathcal{G}^n_d$ denote the set of d-regular graphs on n vertices. Firstly, the Kahn-Lovász Theorem (see e.g. [2]) states that a graph with degree sequence $r_1, \ldots, r_n$ has at most $\prod_{i=1}^n (r_i!)^{1/2r_i}$ perfect matchings. In particular, an n-vertex d-regular graph has at most $(d!)^{n/2d}$ perfect matchings. To determine an upper bound for the number of 1-factorizations of a d-regular graph G, one can simply apply the Kahn-Lovász Theorem repeatedly to obtain $M(G) \leq \prod_{r=1}^d (r!)^{n/2r}$. Using Stirling's approximation, we obtain the following result.

Theorem 6.10. *Suppose $n \in \mathbb{N}$ is even with $1/n \ll 1$, and $d \geq n/2$. Then every $G \in \mathcal{G}^n_d$ satisfies*

$$M(G) \leq \left(\left(1 + n^{-1/2} \right) \frac{d}{e^2} \right)^{dn/2}.$$

On the other hand, Ferber, Jain, and Sudakov [13] proved the following lower bound for the number of distinct 1-factorizations in dense regular graphs.

Theorem 6.11 ([13, Theorem 1.2]). *Suppose $C > 0$ and $n \in \mathbb{N}$ is even with $1/n \ll 1/C \ll 1$, and $d \geq (1/2 + n^{-1/C})n$. Then every $G \in \mathcal{G}_d^n$ satisfies*

$$M(G) \geq \left(\left(1 - n^{-1/C} \right) \frac{d}{e^2} \right)^{dn/2}.$$

Theorems 6.10 and 6.11 immediately yield the following corollary:

Corollary 6.12. *Suppose $C > 0$ and $n \in \mathbb{N}$ is even with $1/n \ll 1/C \ll 1$, and $d \geq (1/2 + n^{-1/C})n$. Then*

$$\frac{M(G)}{M(H)} \leq \exp\left(2n^{1-1/C} d \right),$$

for all $G, H \in \mathcal{G}_d^n$.

For $G \in \mathcal{G}_{[n-1]}^{\mathrm{col}}$ and a set of colours $D \subseteq [n-1]$, we define the *restriction of G to D*, denoted $G|_D$, to be the spanning subgraph of G containing precisely those edges of G which have colour in D. Observe that $G|_D \in \mathcal{G}_D^{\mathrm{col}}$. We now have all the tools we need to prove Lemma 3.8.

Proof of Lemma 3.8. Let $C > 0$ be the constant given by Corollary 6.12 and suppose that $1/n \ll 1/C, \mu, \varepsilon$. Let $\mathbb{P}$ denote the probability measure for the space corresponding to choosing $\mathbf{G} \in \mathcal{G}_{[n-1]}^{\mathrm{col}}$ uniformly at random. Fix $D \subseteq [n-1]$ such that $|D| = \varepsilon n$, and let $\mathbb{P}_D$ denote the probability measure for the space corresponding to choosing $\mathbf{H} \in \mathcal{G}_D^{\mathrm{col}}$ uniformly at random. Let $\mathcal{G}_D^{\mathrm{bad}}$ denote the set of $H \in \mathcal{G}_D^{\mathrm{col}}$ such that H is not ε-locally edge-resilient. For $H \in \mathcal{G}_D^{\mathrm{col}}$, write N_H for the number of distinct completions of H to an element $G \in \mathcal{G}_{[n-1]}^{\mathrm{col}}$; that is, N_H is the number of 1-factorizations of the complement of H. Then

$$
\begin{aligned}
\mathbb{P}\left[\mathbf{G}|_D \text{ is not } \varepsilon\text{-locally edge-resilient} \right] &= \frac{\sum_{H \in \mathcal{G}_D^{\mathrm{bad}}} N_H}{\sum_{H' \in \mathcal{G}_D^{\mathrm{col}}} N_{H'}} \\
&\leq \mathbb{P}_D\left[\mathbf{H} \in \mathcal{G}_D^{\mathrm{bad}} \right] \cdot \exp\left(2n^{2-1/C} \right) \\
&\leq \exp\left(-\varepsilon^3 n^2 / 2000 \right),
\end{aligned}
$$

where we have used Lemma 6.1 and Corollary 6.12. Then, union bounding over choices of D, we deduce that

$$(6.3) \qquad \mathbb{P}\left[\mathbf{G} \text{ is not } \varepsilon\text{-locally edge-resilient} \right] \leq \binom{n-1}{\varepsilon n} \exp\left(-\frac{\varepsilon^3 n^2}{2000} \right) \leq \exp\left(-\frac{\varepsilon^3 n^2}{4000} \right).$$

Now, fix $x \in V$, and fix $c \in [n-1]$. Choose $F \subseteq [n-1] \setminus \{c\}$ of size $|F| = \mu n$ arbitrarily. Write $F^* := F \cup \{c\}$, and let $\mathbb{P}_{F^*}$ denote the probability measure for the space $\mathcal{S}$ corresponding to choosing $\mathbf{H} \in \mathcal{G}_{F^*}^{\mathrm{col}}$ uniformly at random. Let $\mathcal{P}$ be an equitable (ordered) partition of F into four subsets. Let $A_{F^*}^{(x,c)} \subseteq \mathcal{G}_{F^*}^{\mathrm{col}}$ be the set of $H \in \mathcal{G}_{F^*}^{\mathrm{col}}$ such that H has a $5\mu n/4$-well-spread collection of at least $\mu^4 n^2 / 2^{23}$ (x,c)-absorbing gadgets. Then, considering $A_{F^*}^{(x,c)}$, $\mathcal{Q}_{F^*}^{\mathrm{col}}$, $\widetilde{\mathcal{Q}}_{F^*}^{\mathrm{col}}$ as events in $\mathcal{S}$, observe that

$$
\begin{aligned}
\mathbb{P}_{F^*}\left[\overline{A_{F^*}^{(x,c)}} \right] &\leq \mathbb{P}_{F^*}\left[\widetilde{\mathcal{Q}}_{F^*}^{\mathrm{col}} \right] \mathbb{P}_{F^*}\left[\overline{A_{F^*}^{(x,c)}} \,\Big|\, \widetilde{\mathcal{Q}}_{F^*}^{\mathrm{col}} \right] + \mathbb{P}_{F^*}\left[\overline{\widetilde{\mathcal{Q}}_{F^*}^{\mathrm{col}}} \right] \\
&\overset{(6.1)}{\leq} \mathbb{P}_{F^*}\left[\overline{A_{F^*}^{(x,c)}} \,\Big|\, \widetilde{\mathcal{Q}}_{F^*}^{\mathrm{col}} \right] + \mathbb{P}_{F^*}\left[\overline{\mathcal{Q}_{F^*}^{\mathrm{col}}} \right].
\end{aligned}
$$

Thus, applying Lemma 6.9, Lemma 6.3, and Lemma 6.8, we obtain

$$
\begin{aligned}
\mathbb{P}_{F^*}\left[\overline{A_{F^*}^{(x,c)}} \right] &\leq \mathbb{P}_{F^*}\left[r(\mathbf{H}) \leq \mu^4 n^2 / 2^{23} \,\Big|\, \mathbf{H} \in \widetilde{\mathcal{Q}}_{F^*}^{\mathrm{col}} \right] + \mathbb{P}_{F^*}\left[\mathbf{H} \notin \mathcal{Q}_{F^*}^{\mathrm{col}} \right] \\
&\leq \exp\left(-\frac{\mu^4 n^2}{2^{24}} \right) + \exp\left(-\mu^3 n^2 \right) \leq \exp\left(-\frac{\mu^4 n^2}{2^{25}} \right).
\end{aligned}
$$

Then by Corollary 6.12,

$$
\mathbb{P}\left[\mathbf{G}|_{F^*} \notin A_{F^*}^{(x,c)}\right] = \frac{\sum_{H \in \overline{A_{F^*}^{(x,c)}}} N_H}{\sum_{H' \in \mathcal{G}_{F^*}^{\mathrm{col}}} N_{H'}} \leq \mathbb{P}_{F^*}\left[\mathbf{H} \notin A_{F^*}^{(x,c)}\right] \cdot \exp\left(2n^{2-1/C}\right)
$$

$$
\leq \exp\left(-\frac{\mu^4 n^2}{2^{26}}\right).
$$

In particular, with probability at least $1-\exp(-\mu^4 n^2/2^{26})$, $\mathbf{G}$ has a $5\mu n/4$-well-spread collection of at least $\mu^4 n^2/2^{23}$ (x,c)-absorbing gadgets. Now, union bounding over all vertices $x \in V$ and all colours $c \in [n-1]$, we deduce that

$$
(6.4) \qquad \mathbb{P}\left[\mathbf{G} \text{ is not } \mu\text{-robustly gadget-resilient}\right] \leq n^2 \cdot \exp\left(-\frac{\mu^4 n^2}{2^{26}}\right) \leq \exp\left(-\frac{\mu^4 n^2}{2^{27}}\right).
$$

The result now follows by combining (6.3) and (6.4). $\qquad\qquad\square$

7. Modifications and Corollaries

In this section we show how to derive the n odd case of Theorem 1.3 from the case when n is even. We also show how Theorem 1.3(ii) implies Corollary 1.4.

7.1. A rainbow Hamilton cycle for n odd.
We actually derive the n odd case of Theorem 1.3 from the following slightly stronger version of Theorem 1.3(ii) in the case when n is even.

Theorem 7.1. *If n is even and ϕ is a uniformly random 1-factorization of K_n, then for every vertex v, with high probability, ϕ admits a rainbow cycle containing all of the colours and all of the vertices except v.*

We now argue that our proof of Theorem 1.3 for n even is sufficiently robust to also obtain this strengthening. In particular, we can strengthen Lemma 3.9 so that the absorber does not contain v, since (a)–(c) in Lemma 5.3, (a)–(c) in Lemma 5.4, and (a)–(f) in Lemma 5.5 all hold after deleting v from any part in the absorber partition. The proof of Lemma 3.10 is also sufficiently robust to guarantee that the rainbow path from the lemma does not contain v, but we do not need this strengthening, since we can instead strengthen Proposition 3.5 to obtain a rainbow cycle containing $P' - v$ and all of the colours, as follows. If $v \in V(P')$, then we replace v in P' with a $(V_{\mathrm{flex}}, C_{\mathrm{flex}}, G_{\mathrm{flex}})$-cover by deleting v and adding a $(V_{\mathrm{flex}}, C_{\mathrm{flex}}, G_{\mathrm{flex}})$-cover of w, w', and $\phi(vw)$, where w and w' are the vertices adjacent to v in P'. The remainder of the proof proceeds normally, letting $v_\ell := v$ to ensure $v \notin V(P_1'')$. In this procedure, we need to assume that P' is contained in $(V \setminus V', C \setminus C', G')$ with $\delta/19$-bounded remainder (rather than $\delta/18$), but in Lemma 3.9 we can find a 38γ-absorber, which completes the proof.

Now we show how Theorem 7.1 implies the odd n case of Theorem 1.3.

Proof of Theorem 1.3, n odd case. When n is odd, any optimal edge-colouring of K_n has n colour classes, each containing precisely $(n-1)/2$ edges. For every colour c, there is a unique vertex which has no incident edges of colour c, and for every vertex v, there is a unique colour such that v has no incident edges of this colour. Thus, we can obtain a 1-factorization ϕ' of K_{n+1} from an optimal edge-colouring ϕ of K_n in the following way. We add a vertex z, and for every other vertex v, we add an edge zv, where $\phi'(zv)$ is the unique colour c such that v is not incident to a c-edge in K_n. Note that this operation produces a bijection from the set of n-edge-colourings of K_n to the set of 1-factorizations of K_{n+1}. Thus, if n is odd and ϕ is a uniformly random optimal edge-colouring of K_n, then ϕ' is a uniformly random optimal edge-colouring of K_{n+1}. By Theorem 7.1, with high probability there is a rainbow cycle F in K_{n+1} containing all of the colours and all of the vertices except z, so F is a rainbow Hamilton cycle in K_n, satisfying Theorem 1.3(ii). Deleting any edge from F gives a rainbow Hamilton path, as required in Theorem 1.3(i). $\qquad\square$

7.2. Symmetric Latin squares.

Now we use Theorem 1.3 to prove Corollary 1.4.

Proof of Corollary 1.4. Suppose that $n \in \mathbb{N}$ is odd. Firstly, note that there is a one-to-one correspondence between the set $\mathcal{L}_n^{\mathrm{sym}}$ of symmetric $n \times n$ Latin squares with symbols in $[n]$ (say) and the set Φ_n of optimal edge-colourings of K_n on vertices $[n]$ and with colours in $[n]$. Indeed, let $\phi \in \Phi_n$. Then we can construct a unique symmetric Latin square $L_\phi \in \mathcal{L}_n^{\mathrm{sym}}$ by putting the symbol $\phi(ij)$ in position (i,j) for all edges $ij \in E(K_n)$, and for each position (i,i) on the leading diagonal we now enter the unique symbol still missing from row i. Conversely, let $L \in \mathcal{L}_n^{\mathrm{sym}}$. We can obtain a unique element $\phi_L \in \Phi_n$ from L in the following way. Colour each edge ij of the complete graph K_n on vertex set $[n]$ with the symbol in position (i,j) of L. It is clear that ϕ_L is proper, and thus ϕ_L is optimal. Moreover, it is clear that we can uniquely recover L from ϕ_L.

Now, let K_n° be the graph obtained from K_n by adding a loop ii at every vertex $i \in [n]$, and for every $\phi \in \Phi_n$, let ϕ° be the unique proper n-edge-colouring of K_n° such that the restriction of ϕ° to the underlying simple graph is ϕ. The rainbow 2-factors in K_n° admitted by ϕ° correspond to transversals in L_ϕ in the following way. If $L \in \mathcal{L}_n^{\mathrm{sym}}$ and T is a transversal of L, then the subgraph of K_n° induced by the edges ij where $(i,j) \in T$ is a rainbow 2-factor. If σ is the underlying permutation of T, then the cycles of this rainbow 2-factor are precisely the cycles in the cycle decomposition of σ, up to orientation. Therefore a rainbow Hamilton cycle in K_n° corresponds to two disjoint Hamilton transversals in L_ϕ.

By these correspondences, for n odd, if $\mathbf{L} \in \mathcal{L}_n^{\mathrm{sym}}$ is a uniformly random symmetric $n \times n$ Latin square, then $\phi_{\mathbf{L}}$ is a uniformly random optimal edge-colouring of K_n. By Theorem 1.3(ii), $\phi_{\mathbf{L}}$ admits a rainbow Hamilton cycle F with high probability. Since F is also a rainbow Hamilton cycle in K_n°, the corresponding transversals in $\mathbf{L}$ are Hamilton, as desired. □

Note that, if n is odd, the leading diagonal of any $L \in \mathcal{L}_n^{\mathrm{sym}}$ is also a transversal, disjoint from any Hamilton transversal. Indeed, by symmetry all symbols appear an even number of times off of the leading diagonal, and therefore an odd number of times (and thus exactly once) on the leading diagonal.

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